Stability Analysis of Discrete-Time Polynomial Fuzzy-Model-Based Control Systems With Time Delay and Positivity Constraints Through Piecewise Taylor Series Membership Functions

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***Abstract*—This article proposes a novel membership-function dependent approach for the stability/positivity investigation and controller design of a nonlinear, discrete-time system with time delay which is represented by a polynomial fuzzy model to enhance the accuracy of the approximation. The polynomial fuzzy controller is designed based on imperfect premise matching design concept and a set of sum of square (SOS)-based conditions is formulated to check the positivity and stability of the control system. To relax the conservative SOS-based conditions, we employ piecewise Taylor series membership functions (PTSMFs) and introduce the membership function knowledge into the stability conditions. Slack matrices are used to represent the membership function and premise variable knowledge, which contains: 1) regional approximation error between PTSMFs and original membership functions; 2) regional bounds of PTSMFs; and 3) the property knowledge of interpolation membership function of PTSMFs and regional bounds of premise variables. A simulation example is finally given to verify our novel stability/positivity conditions and discrete-time polynomial fuzzy (DPF) controller design.**

***Index Terms*—Discrete-time polynomial fuzzy (DPF) control systems, imperfect premise match design, piecewise Taylor series membership functions (PTSMFs), stability and positivity analysis, sum of squares (SOSs), time delay.**

I. Introduction

POSITIVE systems, referring to one type of systems whose system states always work in positive space with non-negative initial condition [1]–[4]. The theoretical study of positive systems has become important since the dynamical process in chemical reactor, heat exchanges, and storage systems can be effectively modeled with systems whose state variables are non-negative. However, such processes are always involved with time delays which can be a root cause of poor performance and even instability in such systems [5], [6]. Therefore, any realistic dynamical positive model must include the time delay into stability investigation and control synthesis of positive nonlinear systems.

For now, different efficient control methods have been employed to control nonlinear system, such as neural control [7], [8], fuzzy control [9]–[14], and so on. To model a discrete-time positive nonlinear system with time delay, the Takagi–Sugeno (T–S) fuzzy model [15], [16], has been widely used as a valid method to describe the positive nonlinear plant by a combination of local linear subsystems summed up through membership functions under traditional sector nonlinearity technique [17], [18]. The advantage of T–S fuzzy approximation is that a T–S fuzzy model is a convex combination of a set of local linear subsystems. As a result, all the linear methods can be applicable to a nonlinear system represented by a T–S fuzzy model [19]. Another advantage is that positivity and stability conditions can be formulated as linear matrix inequalities (LMIs) conditions which can be solved numerically by convex technique [15], [20]. In the design of a T–S model-based fuzzy controller, perfect premise matching design concept is typically used [15], [16], [21], [22] where the fuzzy controller must share the same number of rules and membership functions with the T–S fuzzy model. In this way, the feedback gains of the controller are obtained by using the LMI stability and positivity conditions instead of predefined design methods such as pole placement. However, this may impose constraints on designing controllers (a large number of fuzzy rules).

Recently, the polynomial fuzzy models gained a considerable attention since they are able to describe nonlinear systems by a set of local polynomial subsystems weighted by membership functions through Taylor series expansion [23]–[26]. However, so far, there has been only an attempt [27] to use the polynomial fuzzy model for modeling a discrete-time

Manuscript received August 19, 2019; revised November 30, 2019; accepted January 18, 2020. Date of publication February 10, 2020; date of current version November 18, 2021. This work was supported in part by the Queen Mary University of London, in part by the Natural Science Foundation of Hebei Province under Project F2019203505, and in part by the Chinese Scholarship Council. This article was recommended by Associate Editor X. Zhao. *(Corresponding author: Xiaomiao Li.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TSMC.2020.2969095.

Digital Object Identifier 10.1109/TSMC.2020.2969095

positive nonlinear system with time delay. Polynomial models substantially improve the approximation capability compared with basic T–S fuzzy models as they need a smaller number of fuzzy rules by keeping the nonlinear polynomial terms. Moreover, positivity and stability conditions for polynomial fuzzy control design can be numerically formulated using more powerful techniques called the sum of squares (SOSs) instead of LMIs [28]–[31]. To remove the constraints imposed by perfect premise matching design concept, we use imperfect premise matching design concept [23], [32] to allow the number of rules and membership functions of the controller and the ones of the model to be different. In this way, the flexibility designing controller is improved. However, as employing imperfect premise matching design concept, cross terms of membership functions which can group Lyapunov inequalities in perfect premise matching design cannot be applied anymore, therefore, it leads to the conservativeness of stability analysis.

To relax the stability conditions for positive discrete-time polynomial fuzzy (DPF) control systems with time delay based on imperfect premise matching design concept, there are two methods that we can follow. One is to attempt various kinds of Lyapunov function candidates, and the other is introducing the membership function knowledge via slack matrices into the stability analysis. In the first direction, quadratic Lyapunov–Krasovskii function is frequently utilized in the literature to formulate the stability conditions [16], [33]. A new form of Lyapunov function candidate, i.e., co-positive Lyapunov–Krasovskii function (CLKF), is also proposed [20], [34] which consider the property of a positive nonlinear system to relax a conservative stability formulation. The major advantage of CLKF are that the resulting stability conditions is time-delay independent [4], [16]. The other approach, where the membership function knowledge is employed to relax the stability conditions [23], [24], [32], [35]–[37], a limited work can be found toward relaxing the conservative formulation for continuous positive polynomial fuzzy systems with time delay [25], [26]. In [25], the membership function knowledge is considered through piecewise linear membership functions, and in [26], the membership functions are represented as symbol variables carrying the positive grade of membership functions into the stability analysis. As for positive DPF systems with time delay, there is only an attempt in the work [27] to relax the stability analysis by approximated membership functions.

In this article, inspired by the works in [35], [38], and [39], we propose to employ piecewise Taylor series membership functions (PTSMFs) for the stability/positivity analysis of DPF closed-loop systems with time delay. Compared with piecewise linear membership functions which estimate the grades of membership based on linear interpolation technique [25], [27], PTSMFs obtain polynomial functions based on Taylor series expansion corresponding to predetermined sample points to replace the original grades of membership. Therefore, PTSMFs are able to show high approximation accuracy where the approximation error between PTSMFs and the original membership functions depend on the Taylor series expansion degree and the gap of sample points. To further relax stability conditions, instead of introducing global membership function and premise variable knowledge among the overall state space [25], [27], [35], we propose to introduce new information into the stability analysis. This information includes: 1) the regional bounds of PTSMFs; 2) the property knowledge of interpolation membership function of PTSMFs and regional bounds of premise variables corresponding to each substate region; and 3) the regional approximation error between PTSMFs and original membership functions. In this way, additional information can be brought into stability and positivity investigation and as a result more relaxed stability and positivity conditions can be obtained.

To summarize the contribution of this article as follows.

1) *Imperfect premise matching* method is employed to conduct the stability and positivity analysis of a DPF closed-loop system with time delay,
2) *PTSMFs* will be employed to bring membership function knowledge, i.e., the regional approximated error of PTSMFs into the stability analysis,
3) With employing PTSMFs, slack matrices carry the regional bounds of PTSMFs, the property knowledge of interpolation membership function of PTSMFs and regional bounds of premise variables on stability analysis to further improve the relaxedness of stability conditions.

The remaining structure of this article is organized as follows. Section II gives the descriptions of DPF model and controller based on the imperfect premise matching design concept. In Section III, membership-function-dependent stability and positivity conditions is proposed which contain the regional approximated error between PTSMFs and original ones, the regional bounds of PTSMFs, the property knowledge of interpolation membership function of PTSMFs and bounds of premise variables. In Section IV, a simulation example is provided to demonstrate the merit of the proposed stability/positivity theorem. Section V provided a conclusion.

II. Notations and Preliminaries

A. Notation

The following notations are employed in this article.

1) One term $x_1^{f_1}(\rho)x_2^{f_2}(\rho)\dots x_n^{f_n}(\rho)$ with degree $f = \sum_{i=1}^{n} f_i$ is defined as monomial of $\mathbf{x}(\rho) = [x_1(\rho), x_2(\rho), \dots, x_n(\rho)]$, where f_i, $i \in \{1, 2, \dots, n\}$ a non-negative integer.
2) Polynomial $\mathbf{q}(\mathbf{x}(\rho))$ is considered as an expression with a linear combination of monomials by real coefficients. Polynomial $\mathbf{q}(\mathbf{x}(\rho))$ is defined as an SOS if it can be represented by $\mathbf{q}(\mathbf{x}(\rho)) = \sum_{j=1}^{m} \mathbf{p}_j(\mathbf{x}(\rho))^2$, where $\mathbf{p}_j(\mathbf{x}(\rho))$ is a monomial and $m \in \mathbb{Z}^+$. Obviously, SOS formulation implying $\mathbf{q}(\mathbf{x}(\rho)) \geq 0$.
3) $\mathbf{Y} \succ 0$, $\mathbf{Y} \succcurlyeq 0$, $\mathbf{Y} \prec 0$, and $\mathbf{Y} \preccurlyeq 0$ denote positivity, semipositivity, negativity, and seminegativity for each element of the matrix $\mathbf{Y}$, respectively.
4) $\underline{n} = \{1, 2, \dots, n\}$, $\underline{s} = \{1, 2, \dots, s\}$, and $\underline{j} = \{1, 2, \dots, j\}$ represent the system state number of fuzzy model, the rule number of fuzzy model, and the rule number of fuzzy controller,

respectively, where $n, s, j \in \mathbb{Z}^+$. $\underline{l} = \{1, 2, \ldots, l\}$, where $l \in \mathbb{Z}^+$ is the time delay. $\mathbf{w}(\mathbf{x}(\rho)) = [w_1(\mathbf{x}(\rho)), w_2(\mathbf{x}(\rho)), \ldots, w_d(\mathbf{x}(\rho))], d \in \underline{s}$ and $\mathbf{m}(\mathbf{x}(\rho)) = [m_1(\mathbf{x}(\rho)), m_2(\mathbf{x}(\rho)), \ldots, m_c(\mathbf{x}(\rho))], c \in \underline{j}$ are defined as the normalized membership functions of fuzzy model and fuzzy controller, respectively.

B. DPF Model With Time Delay

Now, the description of the discrete-time nonlinear plant with time delay is given by s rules polynomial fuzzy model. The dth rule subsystem formulation is provided as

$$\begin{aligned}
&\text{Rule } d: \text{ IF } f_1(\mathbf{x}(\rho)) \text{ is } M_1^d \text{ AND} \cdots \text{AND } f_\Psi(\mathbf{x}(\rho)) \text{ is } M_\Psi^d \\
&\text{THEN } \mathbf{x}(\rho+1) = \mathbf{A}_{d0}(\mathbf{x}(\rho))\mathbf{x}(\rho) \\
&\qquad + \sum_{i=1}^{l} \mathbf{A}_{di}\mathbf{x}(\rho - \tau_i) + \mathbf{B}_d(\mathbf{x}(\rho))\mathbf{u}(\rho) \quad (1)\\
&\mathbf{x}(\rho) = \mathbf{\Upsilon}(\rho), \quad \rho = [-\tau_{\max}, 0] \quad (2)
\end{aligned}$$

where fuzzy set is defined as M_q^d in rule d for function $f_q(\mathbf{x}(\rho))$, $q = \{1, 2, \ldots, \Psi\}$, $\Psi \in \mathbb{Z}^+$; $\mathbf{\Upsilon}(\rho)$ is the initial condition of system states; $\mathbf{x}(\rho) \in \Re^n$, $\mathbf{u}(\rho) \in \Re^m$, $\mathbf{A}_{d0}(\mathbf{x}(\rho)) \in \Re^{n\times n}$, $\mathbf{A}_{di} \in \Re^{n\times n}$, and $\mathbf{B}_d(\mathbf{x}(\rho)) \in \Re^{n\times m}$ are known state vector, control input vector, polynomial system, time delay, and input matrices, respectively. τ_i, $i \in \underline{l} = \{1, 2, \ldots, l\}$, is the time delay. The system dynamics is described as

$$\mathbf{x}(\rho+1) = \sum_{d=1}^{s} w_d(\mathbf{x}(\rho))\left(\mathbf{A}_{d0}(\mathbf{x}(\rho))\mathbf{x}(\rho) + \sum_{i=1}^{l}\mathbf{A}_{di}\mathbf{x}(\rho-\tau_i) + \mathbf{B}_d(\mathbf{x}(\rho))\mathbf{u}(\rho)\right) \quad (3)$$

where $w_d(\mathbf{x}(\rho))$ is defined as the normalized membership grade and

$$w_d(\mathbf{x}(\rho)) = \frac{\prod_{q=1}^{\Psi} \mu_{M_q^d}(f_q(\mathbf{x}(\rho)))}{\sum_{k=1}^{p}\prod_{q=1}^{\Psi} \mu_{M_q^k}(f_l(\mathbf{x}(\rho)))} \geq 0$$

$$\sum_{d=1}^{s} w_d(\mathbf{x}(\rho)) = 1$$

and $\mu_{M_q^d}(f_q(\mathbf{x}(\rho)))$ is the membership grade for fuzzy term M_q^d.

Definition 1 [20]: The positivity definition of a system means when the non-negative initial condition $\mathbf{\Upsilon}(\cdot) \succcurlyeq 0$, the trajectory of system states $\mathbf{x}(\rho)$ continue work in positive space for all $\rho \succcurlyeq 0$.

Lemma 1 [20]: The open loop DPF system with time delay (3) with $\mathbf{u}(\rho) = \mathbf{0}$ is originally positive when the system matrices $\mathbf{A}_{d0}(\mathbf{x}(\rho)) \succcurlyeq 0$ and time delay matrices $\mathbf{A}_{di} \succcurlyeq 0$.

C. DPF State-Feed-Back Controller

Based on the imperfect premise matching design concept, the cth rule local formulation of polynomial fuzzy controller is provided

$$\begin{aligned}
&\text{Rule } c: \text{ IF } g_1(\mathbf{x}(\rho)) \text{ is } N_1^c \text{ AND} \cdots \text{AND } g_\Omega(\mathbf{x}(\rho)) \text{ is } N_\Omega^c \\
&\qquad \text{THEN } \mathbf{u}(\rho) = \mathbf{G}_c(\mathbf{x}(\rho))\mathbf{x}(\rho) \quad (4)
\end{aligned}$$

where fuzzy set is defined as N_l^c in rule j for function $g_l(\mathbf{x}(\rho))$ and $l = \{1, 2, \ldots, \Omega\}$, $\Omega \in \mathbb{Z}^+$; $\mathbf{G}_c(\mathbf{x}(\rho)) \in \Re^{m\times N}$, $c \in \underline{j}$, represents the polynomial feedback gain of fuzzy controller. The output of polynomial fuzzy controller is defined

$$\mathbf{u}(\rho) = \sum_{c=1}^{j} m_c(\mathbf{x}(\rho))\mathbf{G}_c(\mathbf{x}(\rho))\mathbf{x}(\rho) \quad (5)$$

where $m_c(\mathbf{x}(\rho))$ represents the normalized membership grade and

$$m_c(\mathbf{x}(\rho)) = \frac{\prod_{l=1}^{\Omega} \mu_{N_l^c}(g_l(\mathbf{x}(\rho)))}{\sum_{k=1}^{c}\prod_{l=1}^{\Omega} \mu_{N_l^k}(g_l(\mathbf{x}(\rho)))} \geq 0$$

$$\sum_{c=1}^{j} m_c(\mathbf{x}(\rho)) = 1$$

where $\mu_{N_l^c}(g_l(\mathbf{x}(\rho)))$ is defined as the membership grade for fuzzy set N_l^c.

Remark 1: In the imperfect premise matching design concept, we let the membership function $m_j(\mathbf{x}(k))$ of the PFMB controller to be freely chosen as $m_j(\mathbf{x}(k)) \neq w_i(\mathbf{x}(k))$ for any $p \neq c$. Therefore, even though we confront the situation that a large number of rules and/or complicated membership functions fuzzy model, we still can design a smaller number of rules and/or simple membership functions controller. Therefore, the imperfect premise matching design concept offers the greatest design flexibility of designing polynomial fuzzy controller and further reduces the implementation cost.

III. Positivity and Stability Analysis

The DPF closed-loop system with time delay is formed by substituting the polynomial fuzzy controller (5) into the polynomial fuzzy model (3)

$$\mathbf{x}(\rho+1) = \sum_{d=1}^{s}\sum_{c=1}^{j} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho)) \times \left((\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \mathbf{B}_d(\mathbf{x}(\rho))\mathbf{G}_c(\mathbf{x}(\rho)))\mathbf{x}(\rho) + \sum_{i=1}^{l}\mathbf{A}_{di}\mathbf{x}(\rho-\tau_i)\right). \quad (6)$$

Assumption 1: $\mathbf{A}_{di} \succcurlyeq 0$ should be satisfied for the polynomial fuzzy model (3), otherwise, no polynomial fuzzy controller can be founded to guarantee the positivity of DPF closed-loop system with time delay.

Lemma 2 [20]: For the DPF closed-loop system with time delay (6), positivity for $\mathbf{\Upsilon}(\cdot) \succcurlyeq 0$ means $\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \mathbf{B}_d(\mathbf{x}(\rho))\mathbf{G}_c(\mathbf{x}(\rho)) \succcurlyeq 0$ and $\mathbf{A}_{di} \succcurlyeq 0$.

Remark 2: Based on the imperfect premise matching design concept, the shape polynomial fuzzy controller membership function $m_c(\mathbf{x}(\rho))$ and fuzzy model membership function $w_d(\mathbf{x}(\rho))$ satisfy $m_c(\mathbf{x}(\rho)) \neq w_d(\mathbf{x}(\rho))$ for $c \neq d$.

Remark 3: A great deal of work has been done in [33] to ease the stability and positivity analysis for the nonlinear system using duality principals and dual system equivalence of stability. Since the stability conditions between two systems under duality are equivalent, we transfer the original system to a dual equivalent system whose matrices are transposed compared with the original system. The advantage of doing so is that using a dual system equivalent, we can get an integrated formulation and cancelation of time delay during the investigation of Lyapunov stability analysis.

The dual system formulation of (6) is demonstrated as follows:

$$\begin{aligned}\mathbf{x}(\rho+1) = &\sum_{d=1}^{s}\sum_{c=1}^{j} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))\\ &+ \mathbf{B}_d(\mathbf{x}(\rho))\mathbf{G}_c(\mathbf{x}(\rho))^T\mathbf{x}(\rho)\\ &\times\left(\left(\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \sum_{i=1}^{l}\mathbf{A}_{di}^T\mathbf{x}(\rho-\tau_i)\right)\right).\end{aligned} \tag{7}$$

The equivalence of stability conditions for the original system and its dual system is proved in the following.

Proof: The nonlinear DPF closed-loop system with time delay (6) can be re-expressed by

$$\begin{aligned}\mathbf{x}(\rho+1) = &\sum_{d=1}^{s}\sum_{c=1}^{j}\sum_{i=0}^{l} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))\\ &\times(\mathbf{E}_{dci}(\mathbf{x}(\rho)\mathbf{x}(\rho-\tau_i)))\end{aligned} \tag{8}$$

where $\tau_0 = 0$, for $i = 0, \mathbf{E}_{dc0}(\mathbf{x}(\rho)) = (\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \mathbf{B}_d(\mathbf{x}(\rho))\mathbf{G}_c(\mathbf{x}(\rho)))$, for $i = 1,\ldots,l, \mathbf{E}_{dci} = \mathbf{A}_{di}$, (8) can be re-expressed by

$$\begin{aligned}\mathbf{x}(\rho+1) = &\sum_{d=1}^{s}\sum_{c=1}^{j}\sum_{i=0}^{l}\\ &\times w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))(\mathbf{E}_{dci}(\mathbf{x}(\rho)\mathbf{x}(\rho-\tau_i))\\ = &\sum_{d_\rho=1}^{s}\sum_{c_\rho=1}^{j}\sum_{i_\rho=1}^{l}\sum_{d_{\rho-1}=1}^{s}\sum_{c_{\rho-1}=1}^{j}\sum_{i_{\rho-1}=1}^{l}\cdots\\ &\times\sum_{d_0=1}^{s}\sum_{c_0=1}^{j}\sum_{i_0=1}^{l} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))\\ &\times w_{d_{\rho-1}}(\mathbf{x}(\rho-1))m_{c_{\rho-1}}(\mathbf{x}(\rho-1))\cdots\\ &\times w_{d_0}(\mathbf{x}(0))m_{c_0}(\mathbf{x}(0))\\ &\times\Big(\mathbf{E}_{dci}\big(\mathbf{x}(\rho)\times\mathbf{E}_{d_{\rho-1}c_{\rho-1}i_{\rho-1}}(\mathbf{x}(\rho-1))\ \cdots\\ &\quad\times\mathbf{E}_{d_0c_0i_0}(\mathbf{x}(0))\times\mathbf{x}\big(-\tau_i-\tau_{i_{\rho-1}}-\cdots\tau_{i_0}\big)\Big).\end{aligned} \tag{9}$$

For the dual system (7), we can rewrite the term as

$$\begin{aligned}\mathbf{x}(\rho+1) = &\sum_{d=1}^{s}\sum_{c=1}^{j}\sum_{i=0}^{l} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))\\ &\times\big(\mathbf{E}_{dci}^T(\mathbf{x}(\rho)\mathbf{x}(\rho-\tau_i))\big).\end{aligned} \tag{10}$$

Equation (10) can be re-expressed by

$$\begin{aligned}\mathbf{x}(\rho+1) = &\sum_{d=1}^{s}\sum_{c=1}^{j}\sum_{i=0}^{l} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))\\ &\times\big(\mathbf{E}_{dci}^T(\mathbf{x}(\rho)\mathbf{x}(\rho-\tau_i))\big)\\ = &\sum_{d_\rho=1}^{s}\sum_{c_\rho=1}^{j}\sum_{i_\rho=1}^{l}\sum_{d_{\rho-1}=1}^{s}\sum_{c_{\rho-1}=1}^{j}\\ &\times\sum_{i_{\rho-1}=1}^{l}\cdots\times\sum_{d_0=1}^{s}\sum_{c_0=1}^{j}\sum_{i_0=1}^{l} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))\\ &\times w_{d_{\rho-1}}(\mathbf{x}(\rho-1))m_{c_{\rho-1}}(\mathbf{x}(\rho-1))\cdots\\ &\times w_{d_0}(\mathbf{x}(0))m_{c_0}(\mathbf{x}(0))\\ &\times\Big(\mathbf{E}_{dci}^T\Big(\mathbf{x}(\rho)\times\mathbf{E}_{d_{\rho-1}c_{\rho-1}i_{\rho-1}}^T(\mathbf{x}(\rho-1))\cdots\\ &\quad\times\mathbf{E}_{d_0c_0i_0}^T(\mathbf{x}(0))\times\mathbf{x}\big(-\tau_i-\tau_{i_{\rho-1}}-\cdots\tau_{i_0}\big)\Big).\end{aligned} \tag{11}$$

Assuming the dual system (11) is stable with initial condition $\mathbf{x}(0) = \Upsilon(\mathbf{0}) \succcurlyeq 0$ and $\mathbf{x}(-\tau_i-\tau_{i_{\rho-1}}-\cdots\tau_{i_0}) = \Upsilon(-\tau_i-\tau_{i_{\rho-1}}-\cdots\tau_{i_0}) \succcurlyeq 0$, then $\mathbf{x}(\rho+1)\to 0$ as $\rho\to\infty$. Therefore, we can conclude that

$$\begin{aligned}&\mathbf{E}_{dci}^T\Big(\mathbf{x}(\rho)\times\mathbf{E}_{d_{\rho-1}c_{\rho-1}i_{\rho-1}}^T(\mathbf{x}(\rho-1))\cdots\Big)\\ &\quad\times\mathbf{E}_{d_0c_0i_0}^T(\mathbf{x}(0))\to 0.\end{aligned} \tag{12}$$

Considering (12), we can rewrite the term as

$$\begin{aligned}&\big(\mathbf{E}_{d_0c_0i_0}(\mathbf{x}(0))\times\mathbf{E}_{d_{\rho-1}c_{\rho-1}i_{\rho-1}}(\mathbf{x}(\rho-1))\\ &\quad\times\ \mathbf{E}_{dci}(\mathbf{x}(\rho))\big)^T\to 0\end{aligned} \tag{13}$$

and (13) implies to

$$\mathbf{E}_{d_0c_0i_0}(\mathbf{x}(0))\times\mathbf{E}_{d_{\rho-1}c_{\rho-1}i_{\rho-1}}(\mathbf{x}(\rho-1))\times\mathbf{E}_{dci}(\mathbf{x}(\rho))\to 0. \tag{14}$$

As can be seen from the terms (14) and (8), the polynomial fuzzy controller (5) based on the imperfect premise matching method can stabilize the original system (6) as well as its dual equivalent controller system in (7). ■

A. Positivity Analysis

Remark 4: Based on Lemma 2, the positivity SOS-based conditions for DPF closed-loop system with time delay are summarized from (32) to (34) in Theorem 1.

B. Piecewise Taylor Series Membership Function

To relax the stability conditions for the DPF closed-loop system with time delay, different methods of constructing approximated membership functions, such as staircase membership function and piecewise linear membership function have been utilized to include membership function knowledge to the stability analysis [25], [40]. As mentioned in Section I, to reduce the approximation error, we propose to employ PTSMFs here to relax the stability conditions. Since we use Taylor series expansion polynomial functions to construct the PTSMFs, first, the certain number of sample points are chosen from membership functions. Then Taylor series expansion is employed to get a corresponding polynomial function at every sample point. For every two adjacent sample points, one interpolation function is designed to connect with Taylor series expansion polynomial functions. Finally, PTSMFs are obtained to approximate the original membership functions.

The construction process of PTSMFs will be demonstrated numerically in the following. Considering a nonlinear systems with system states $\mathbf{x}(\rho) = [x_1(\rho), x_2(\rho), \ldots, x_n(\rho)]$ and $\mathbf{x}(\rho) \in \delta$, $\delta \in \Re^n$ is overall system state space. For the ϖth system state $x_\varpi(\rho)$, the corresponding region is partitioned e_ϖ connected subregions by sample points. We then define the overall state space such that it is partitioned e connected subregions in the state space denoted as δ_g, $g = 1, 2, \ldots, e$. Based on above definition, define the whole state space with subregions $\delta = \cup_{g=1}^{e} \delta_g$, $e = \prod_{\varpi=1}^{n} e_\varpi$. Since $x_\varpi(\rho)$ is confined to the e_ϖ subregion in the range of adjacent sample points $x_{\varpi 1g}$ to $x_{\varpi 2g}$, we can construct each subregion δ_g possesses 2^n end points and $\prod_{\varpi=1}^{n}(e_\varpi + 1)$ sample points for overall state space.

We define $h_{dc}(\mathbf{x}(\rho)) = w_d(\mathbf{x}(\rho)) m_c(\mathbf{x}(\rho))$ where $w_d(\mathbf{x}(\rho))$ and $m_c(\mathbf{x}(\rho))$ are original membership functions of the polynomial fuzzy model and controller, respectively. The formation of PTSMFs $\hat{h}_{dcg}(\mathbf{x}(\rho))$ when $\mathbf{x}(\rho) \in \delta_g$ is given as

$$\hat{h}_{dcg}(\mathbf{x}(\rho)) = \sum_{q_1=1}^{2} \sum_{q_2=1}^{2} \cdots \sum_{q_n=1}^{2} \prod_{\varpi=1}^{n} \varrho_{\varpi q_\varpi g}(x_\varpi(\rho)) \times \varphi_{dcq_1 q_2 \cdots q_n g}(\mathbf{x}(\rho)) \tag{15}$$

where $\varphi_{dcq_1 q_2 \cdots q_n g}(\mathbf{x}(\rho))$ is Taylor series expansion polynomial function for original membership function $h_{dc}(\mathbf{x}(\rho))$ corresponding to sample points $x_{\varpi q_\varpi g}$. The term $\varrho_{\varpi q_\varpi g}(x_\varpi(\rho))$ serves as a function of connecting adjacent two Taylor series expansion polynomial functions together satisfies $0 < \varrho_{\varpi q_\varpi g}(x_\varpi(\rho)) < 1$ and $\varrho_{\varpi 1g}(x_\varpi(\rho)) + \varrho_{\varpi 2g}(x_\varpi(\rho)) = 1$.

In the process of Taylor series expansion polynomial function $\varphi_{dcq_1 q_2 \cdots q_n g}(\mathbf{x}(\rho))$, the formulation of Taylor series expansion is first considered

$$n(\mathbf{x}(\rho)) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{\varpi=1}^{n} (x_\varpi(\rho) - x_{\varpi 0}) \frac{\partial}{\partial x_\varpi(\rho)} \right)^k \times n(\mathbf{x}(\rho))|_{(x_\varpi(\rho) = x_{\varpi 0}, \varpi = 1,2,\ldots,n)} \tag{16}$$

where $n(\mathbf{x}(\rho))$ represents an arbitrary function of $\mathbf{x}(\rho)$, $[(\partial^k n(\mathbf{x}(\rho)))/(\partial x_\varpi(\rho))]$ is the constant value by conducting kth partial derivative of $n(\mathbf{x}(\rho))$ corresponding to $x_\varpi(\rho)$ and replacing variable $x_\varpi(\rho)$ with expansion points $x_{\varpi 0}$, $\varpi = 1, 2, \ldots, n$.

Based on the Taylor series expansion technique introduced above, Taylor expansion for original membership function $h_{dc}(\mathbf{x}(\rho)) = w_d(\mathbf{x}(\rho)) m_c(\mathbf{x}(\rho))$ on the sample points $x_{\varpi q_\varpi g}$ can be described by

$$\varphi_{dcq_1 q_2 \cdots q_n g}(\mathbf{x}(\rho)) = \sum_{k=0}^{\tau-1} \frac{1}{k!} \left(\sum_{\varpi=1}^{n} \left(x_\varpi(\rho) - x_{\varpi q_\varpi g} \right) \frac{\partial}{\partial x_\varpi(\rho)} \right)^k \times h_{dc}(\mathbf{x}(\rho))|_{(x_\varpi(\rho) = x_{\varpi q_\varpi g}, \varpi = 1,2,\ldots,n)} \tag{17}$$

where τ is the Taylor series expansion degree, meaning the Taylor series expansion polynomial function $\varphi_{q_1 q_2 \cdots q_n g}(\mathbf{x}(\rho))$ is obtained with $\tau - 1$ degree of state variables $\mathbf{x}(\rho)$.

$\varrho_{\varpi q_\varpi g}(x_\varpi(\rho))$ is used to connect adjacent sample points $x_{\varpi 1g}$ and $x_{\varpi 2g}$. Based on interpolation technique, when $\mathbf{x}(\rho) \in \delta_g$, we have following relationship:

$$\frac{\varphi_{dcq_1 q_2 \cdots 2 \cdots q_n g}(\mathbf{x}(\rho)) - \hat{h}_{dcg}(\mathbf{x}(\rho))}{x_{\varpi 2g} - x_\varpi(\rho)} = \frac{\varphi_{dcq_1 q_2 \cdots 2 \cdots q_n g}(\mathbf{x}(\rho)) - \varphi_{dcq_1 q_2 \cdots 1 \cdots q_n g}(\mathbf{x}(\rho))}{x_{\varpi 2g} - x_{\varpi 1g}} \tag{18}$$

where we choose $q_\varpi = 1$ and $q_\varpi = 2$ for $\varphi_{dcq_1 q_2 \cdots 1 \cdots q_n g}(\mathbf{x}(\rho))$ and $\varphi_{dcq_1 q_2 \cdots 2 \cdots q_n g}(\mathbf{x}(\rho))$, respectively.

Comparing (18) with (15), we can get

$$\varrho_{\varpi 1g}(x_\varpi(\rho)) = \frac{x_{\varpi 2g} - x_\varpi(\rho)}{x_{\varpi 2g} - x_{\varpi 1g}}$$
$$\varrho_{\varpi 2g}(x_\varpi(\rho)) = 1 - \varrho_{\varpi 1g}(x_\varpi(\rho)). \tag{19}$$

Now, PTSMFs are obtained by substituting (17) and (19) into (15).

C. SOS-Based Stability Analysis

Subject to positivity conditions based on Remark 4, the following polynomial CLKF is employed to investigate the stability of (7)

$$V(\mathbf{x}(\rho)) = \mathbf{x}^T(\rho)\boldsymbol{\eta} + \sum_{m=1}^{s} \sum_{i=1}^{l} \sum_{q=1}^{\tau_l} \left(\mathbf{x}(\rho - q)^T \mathbf{A}_{mi} \boldsymbol{\eta} \right) \tag{20}$$

where $\boldsymbol{\eta} = [\eta_1, \eta_2, \ldots, \eta_n]^T \succ 0$.

Considering the terms (7) and (20), we have

$$\begin{aligned} \Delta V(\mathbf{x}(\rho)) &= V(\mathbf{x}(\rho+1)) - V(\mathbf{x}(\rho)) \\ &= \sum_{d=1}^{s} \sum_{c=1}^{j} w_d(\mathbf{x}(\rho)) m_c(\mathbf{x}(\rho)) \\ &\quad \times \Bigg(\mathbf{x}^T(\rho)(\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \mathbf{B}_d(\mathbf{x}(\rho)) \mathbf{G}_c(\mathbf{x}(\rho))) \\ &\qquad + \mathbf{x}^T(\rho - \tau_i) \sum_{i=1}^{l} \mathbf{A}_{di} \Bigg) \boldsymbol{\eta} \\ &\quad + \sum_{m=1}^{s} \sum_{i=1}^{l} \left(\mathbf{x}(\rho)^T \mathbf{A}_{mi} - \mathbf{x}(\rho - \tau_i)^T \mathbf{A}_{mi} \right) \boldsymbol{\eta} \\ &\quad - \mathbf{x}^T(\rho)\boldsymbol{\eta}. \end{aligned} \tag{21}$$

Since the membership functions of the polynomial fuzzy model and fuzzy controllers $w_d(\mathbf{x}(\rho))$ and $m_c(\mathbf{x}(\rho))$ satisfy the property $0 \le w_d(\mathbf{x}(\rho)) \le 1$ and $0 \le m_c(\mathbf{x}(\rho)) \le 1$ for $d \in \underline{s}$, $c \in \underline{j}$, therefore, we get

$$\begin{aligned} \Delta V(\mathbf{x}(\rho)) \le & \sum_{d=1}^{s} \sum_{c=1}^{j} w_d(\mathbf{x}(\rho)) m_c(\mathbf{x}(\rho)) \\ & \times \left(\mathbf{x}^T(\rho)(\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \mathbf{B}_d(\mathbf{x}(\rho)) \mathbf{G}_c(\mathbf{x}(\rho))) \right) \boldsymbol{\eta} \\ & + \sum_{d=1}^{s} \sum_{i=1}^{l} \mathbf{x}^T(\rho - \tau_i) \mathbf{A}_{di} \boldsymbol{\eta} - \mathbf{x}^T(\rho)\boldsymbol{\eta} \\ & + \sum_{m=1}^{s} \sum_{i=1}^{l} \left(\mathbf{x}(\rho)^T \mathbf{A}_{mi} - \mathbf{x}(\rho - \tau_i)^T \mathbf{A}_{mi} \right) \boldsymbol{\eta} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{d=1}^{s}\sum_{c=1}^{j} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho)) \\
&\quad\times \left(\mathbf{x}^T(\rho)(\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \mathbf{B}_d(\mathbf{x}(\rho))\mathbf{G}_c(\mathbf{x}(\rho)))\right)\boldsymbol{\eta} \\
&\quad - \mathbf{x}^T(\rho)\boldsymbol{\eta} \\
&\quad + \sum_{m=1}^{s}\sum_{i=1}^{l} \mathbf{x}(\rho)^T\mathbf{A}_{mi}\boldsymbol{\eta} \\
&= \sum_{d=1}^{s}\sum_{c=1}^{j} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))\mathbf{x}^T(\rho) \\
&\quad\times \Bigg((\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \mathbf{B}_d(\mathbf{x}(\rho))\mathbf{G}_c(\mathbf{x}(\rho)))\boldsymbol{\eta} \\
&\qquad\qquad - \boldsymbol{\eta} + \sum_{m=1}^{s}\sum_{i=1}^{l}\mathbf{A}_{mi}\boldsymbol{\eta}\Bigg) \\
&= \sum_{d=1}^{s}\sum_{c=1}^{j} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))\mathbf{x}^T(\rho) \\
&\quad\times \Bigg(\Bigg(\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \sum_{m=1}^{s}\sum_{i=1}^{l}\mathbf{A}_{mi}\Bigg)\boldsymbol{\eta} \\
&\qquad\qquad + \mathbf{B}_d(\mathbf{x}(\rho))\sum_{w=1}^{n}\Lambda_w^c(\mathbf{x}(\rho)) - \boldsymbol{\eta}\Bigg) \\
&= \sum_{d=1}^{s}\sum_{c=1}^{j} w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho))\mathbf{x}^T(\rho)\mathbf{Q}_{dc}(\mathbf{x}(\rho))
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
\mathbf{Q}_{dc}(\mathbf{x}(\rho)) &= \Bigg(\mathbf{A}_{d0}(\mathbf{x}(\rho)) + \sum_{m=1}^{s}\sum_{i=1}^{l}\mathbf{A}_{mi}\Bigg)\boldsymbol{\eta} \\
&\quad + \mathbf{B}_d(\mathbf{x}(\rho))\sum_{w=1}^{n}\Lambda_w^c(\mathbf{x}(\rho)) - \boldsymbol{\eta} \\
&= \left[q_1^{dc}(\mathbf{x}(\rho)), q_2^{dc}(\mathbf{x}(\rho)), \ldots, q_n^{dc}(\mathbf{x}(\rho))\right]^T
\end{aligned} \tag{23}$$

the polynomial fuzzy controller is

$$\mathbf{G}_c(\mathbf{x}(\rho)) = \left[\frac{\Lambda_1^c(\mathbf{x}(\rho))}{\eta_1}, \frac{\Lambda_2^c(\mathbf{x}(\rho))}{\eta_2}, \ldots, \frac{\Lambda_n^c(\mathbf{x}(\rho))}{\eta_n}\right]$$

$\Lambda_1^c(\mathbf{x}(\rho))$, $\Lambda_2^c(\mathbf{x}(\rho))$, …, $\Lambda_n^c(\mathbf{x}(\rho)) \in \Re^m$ for $c \in \underline{j}$ are to be determined.

Remark 5: Based on Lyapunov stability theory, the asymptotic stability of the dual system (7) can be realized through $V(\rho) > 0$ and $\Delta V(\mathbf{x}(\rho)) < 0$ (not include $\mathbf{x}(\rho) = 0$) which can be obtained through $q_k^{dc}(\mathbf{x}(\rho)) < 0$ for $d \in \underline{s}$, $c \in \underline{j}$, $k \in \underline{n}$.

Remark 6: The positivity and stability of DPF closed-loop system (6) is guaranteed if positivity conditions in Remark 4 and the stability conditions $q_k^{dc}(\mathbf{x}(\rho)) < 0$ for $d \in \underline{s}$, $c \in \underline{j}$, $k \in \underline{n}$ are satisfied together as equivalence of stability between the DPF closed-loop system with time delay (6) and its dual system (7).

Remark 7: The polynomial fuzzy controller can be synthesized to control the closed-loop system (6) positive and stable by solving the basic SOS-based positivity and stability conditions provided by Remark 6. However, the membership function knowledge $w_d(\mathbf{x}(\rho))$ and $m_c(\mathbf{x}(\rho))$ are not considered leading to potentially conservative stability results.

Now, we relax the stability conditions by including the membership function knowledge via: 1) the regional approximation error between original membership functions and PTSMFs; 2) the regional bounds of PTSMFs; and 3) the property knowledge of interpolation membership function of PTSMFs and bounds of premise variables.

First, the membership function knowledge is introduced into stability analysis by the regional approximation error between the original membership functions and PTSMFs. The approximation error is defined as $\Delta h_{dcg}(\mathbf{x}(\rho)) = w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho)) - \hat{h}_{dcg}(\mathbf{x}(\rho))$ when $\mathbf{x}(\rho) \in \delta_{\underline{g}}$. We also define $\underline{\delta}_{dcg} \leq \Delta h_{dcg}(\mathbf{x}(\rho)) \leq \overline{\delta}_{dcg}$ where $\underline{\delta}_{dcg}$ and $\overline{\delta}_{dcg}$ are the lower and upper bound of approximation error $\Delta h_{dcg}(\mathbf{x}(\rho))$ to be determined. Now using the term (22), we can change the $\Delta V(\mathbf{x}(\rho))$ to

$$\begin{aligned}
&\Delta V(\mathbf{x}(\rho)) \\
&= \mathbf{x}^T(\rho)\Bigg(\sum_{g=1}^{e}\xi_g(\mathbf{x}(\rho))\sum_{d=1}^{s}\sum_{c=1}^{j}\hat{h}_{dcg}(\mathbf{x}(\rho)) \times \mathbf{Q}_{dc}(\mathbf{x}(\rho)) \\
&\quad + \sum_{g=1}^{e}\xi_g(\mathbf{x}(\rho))\sum_{d=1}^{s}\sum_{c=1}^{j}\Big(w_d(\mathbf{x}(\rho))m_c(\mathbf{x}(\rho)) \\
&\quad - \hat{h}_{dcg}(\mathbf{x}(\rho)) + \Big(\underline{\delta}_{dcg} - \underline{\delta}_{dcg}\Big)\Big)\mathbf{Q}_{dc}(\mathbf{x}(\rho))\Bigg) \\
&= \mathbf{x}^T(\rho)\Bigg(\sum_{d=1}^{s}\sum_{c=1}^{j}\sum_{g=1}^{e}\xi_g(\mathbf{x}(\rho))\Big(\hat{h}_{dcg}(\mathbf{x}(\rho)) + \underline{\delta}_{dcg}\Big)\mathbf{Q}_{dc}(\mathbf{x}(\rho)) \\
&\quad + \sum_{d=1}^{s}\sum_{c=1}^{j}\sum_{g=1}^{e}\xi_g(\mathbf{x}(\rho))\Big(\Delta h_{dcg}(\mathbf{x}(\rho)) - \underline{\delta}_{dcg}\Big) \\
&\quad \times \mathbf{Q}_{dc}(\mathbf{x}(\rho))\Bigg)
\end{aligned} \tag{24}$$

where $\xi_g(\mathbf{x}(\rho))$ satisfies $\xi_g(\mathbf{x}(\rho)) = 1, \mathbf{x}(\rho) \in \delta_g$, $g = 1, 2, \ldots, e$, otherwise, $\xi_g(\mathbf{x}(\rho)) = 0$.

If we define slack matrix $0 \preceq \mathbf{Y}_{dcg}(\mathbf{x}(\rho)) = [y_1^{dcg}(\mathbf{x}(\rho)), y_2^{dcg}(\mathbf{x}(\rho)), \ldots, y_n^{dcg}(\mathbf{x}(\rho))]^T \in \Re^n$, and

$$\sum_{g=1}^{e}\xi_g(\mathbf{x}(\rho))\mathbf{Y}_{dcg}(\mathbf{x}(\rho)) - \mathbf{Q}_{dc}(\mathbf{x}(\rho)) \succcurlyeq 0.$$

Then we rewrite the term (24) as

$$\begin{aligned}
&\Delta V(\mathbf{x}(\rho)) \\
&\leq \mathbf{x}^T(\rho)\sum_{g=1}^{e}\xi_g(\mathbf{x}(\rho))\Bigg(\sum_{d=1}^{s}\sum_{c=1}^{j}\Big(\hat{h}_{dcg}(\mathbf{x}(\rho)) + \underline{\delta}_{dcg}\Big)\mathbf{Q}_{dc}(\mathbf{x}(\rho)) \\
&\quad + \sum_{d=1}^{s}\sum_{c=1}^{j}\Big(\overline{\delta}_{dcg} - \underline{\delta}_{dcg}\Big)\mathbf{Y}_{dcg}(\mathbf{x}(\rho))\Bigg).
\end{aligned} \tag{25}$$

Compared to the inclusion of the global approximation error in the stability conditions defined on the entire state space, the regional lower and upper approximation error corresponding to each subregion provides more information from membership function in the stability analysis which leads to a more relaxed and feasible conditions.

Then, in order to obtain more relax the stability condition, the regional lower and upper boundary of PTSMFs corresponding to each subregion δ_g are brought into stability analysis. We define $\underline{\zeta}_{dcg} \le \hat{h}_{dcg}(\mathbf{x}(\rho)) \le \overline{\zeta}_{dcg}$ where $\underline{\zeta}_{dcg}$ and $\overline{\zeta}_{dcg}$ are the regional lower and upper bounds of PTSMFs $\hat{h}_{dcg}(\mathbf{x}(\rho))$ when $\mathbf{x}(\rho) \in \delta_g$. The following inequalities hold:

$$\sum_{g=1}^{e} \xi_g(\mathbf{x}(\rho)) \sum_{d=1}^{s} \sum_{c=1}^{j} \Big(\overline{\zeta}_{dcg} - \hat{h}_{dcg}(\mathbf{x}(\rho))\Big) \mathbf{S}_{dcg}(\mathbf{x}(\rho)) \succcurlyeq 0 \tag{26}$$

$$\sum_{g=1}^{e} \xi_g(\mathbf{x}(\rho)) \sum_{d=1}^{s} \sum_{c=1}^{j} \Big(\hat{h}_{dcg}(\mathbf{x}(\rho)) - \underline{\zeta}_{dcg}\Big) \mathbf{SS}_{dcg}(\mathbf{x}(\rho)) \succcurlyeq 0 \tag{27}$$

where $0 \preccurlyeq \mathbf{S}_{dcg}(\mathbf{x}(\rho)) \in \Re^n$ and $0 \preccurlyeq \mathbf{SS}_{dcg}(\mathbf{x}(\rho)) \in \Re^n$ are polynomial vectors.

In addition to regional lower and upper boundary from PTSMFs, the property knowledge of interpolation membership function of PTSMFs are brought into stability analysis to further reduce the conservativeness of stability conditions. Referring to the expression of interpolation membership function of PTSMFs (19), we know that $\varrho_{\varpi q_\varpi g}(x_\varpi(\rho))$ satisfies $\sum_{g=1}^{e} \xi_g(\mathbf{x}(\rho)) \sum_{q_1=1}^{2} \sum_{q_2=1}^{2} \cdots \sum_{q_n=1}^{2} \prod_{\varpi=1}^{n} \varrho_{\varpi q_\varpi g}(x_\varpi(\rho)) = 1$, therefore, we have

$$\sum_{g=1}^{e} \xi_g(\mathbf{x}(\rho)) \sum_{q_1=1}^{2} \sum_{q_2=1}^{2} \cdots \sum_{q_n=1}^{2} \prod_{\varpi=1}^{n} \varrho_{\varpi q_\varpi g}(x_\varpi(\rho)) \times \big(x_\varpi(\rho) - x_{\varpi q_\varpi g}\big) \mathbf{T}_g(\mathbf{x}(\rho)) = 0 \tag{28}$$

where $\mathbf{T}_g(\mathbf{x}(\rho)) \in \Re^n$ is an arbitrary polynomial vectors.

Other than approximated error and regional boundary knowledge from PTSMFs and property knowledge from interpolation membership function of PTSMFs which can be considered as membership dependent stability analysis, we will introduce the premise variable regional boundary knowledge corresponding to subregion δ_g to further relax the stability analysis. As we defined before, for nonlinear systems with system states $\mathbf{x}(\rho) = [x_1(\rho), x_2(\rho), \ldots, x_n(\rho)]$, the ϖth system state $x_\varpi(\rho)$, the corresponding region is divided into d_ϖ connected subregions by sample points $x_{\varpi q \varpi g}$. We divide the overall state space into e connected subregion which are denoted as δ_g, $g = 1, 2, \ldots, e$. Considering $x_\varpi(\rho)$ which is confined to the d_ϖth subregion under the range from adjacent sample points $x_{\varpi 1g}$ to $x_{\varpi 2g}$, we can have

$$\sum_{g=1}^{e} \xi_g(\mathbf{x}(\rho)) \sum_{\jmath=1}^{n} \big(x_\jmath(\rho) - x_{\jmath 1g}\big)\big(x_{\jmath 2g} - x_\jmath(\rho)\big) \mathbf{U}_{\jmath g}(\mathbf{x}(\rho)) \succcurlyeq 0 \tag{29}$$

where $0 \preccurlyeq \mathbf{U}_{\jmath g}(\mathbf{x}(\rho)) \in \Re^n$ is polynomial vector.

Considering the terms (25) to (29) and (15), we have the following inequality:

$$\begin{aligned}
&\Delta V(\mathbf{x}(\rho)) \\
&\le \mathbf{x}^T(\rho) \sum_{g=1}^{e} \xi_g(\mathbf{x}(\rho)) \sum_{q_1=1}^{2} \\
&\quad \times \sum_{q_2=1}^{2} \cdots \sum_{q_n=1}^{2} \prod_{\varpi=1}^{n} \varrho_{\varpi q_\varpi g}(x_\varpi(\rho)) \\
&\quad \times \sum_{d=1}^{s} \sum_{c=1}^{j} \Big(\Big(\varphi_{dcq_1q_2\cdots q_ng}(\mathbf{x}(\rho)) + \underline{\delta}_{dcg}\Big) \mathbf{Q}_{dc}(\mathbf{x}(\rho)) \\
&\qquad + \Big(\overline{\delta}_{dcg} - \underline{\delta}_{dcg}\Big) \mathbf{Y}_{dcg}(\mathbf{x}(\rho)) \\
&\qquad + \big(\overline{\zeta}_{dcg} - \varphi_{dcq_1q_2\cdots q_ng}(\mathbf{x}(\rho))\big) \mathbf{S}_{dcg}(\mathbf{x}(\rho)) \\
&\qquad + \Big(\varphi_{dcq_1q_2\cdots q_ng}(\mathbf{x}(\rho)) - \underline{\zeta}_{dcg}\Big) \mathbf{SS}_{dcg}(\mathbf{x}(\rho)) \\
&\qquad + \big(x_\varpi(\rho) - x_{\varpi q_\varpi g}\big) \mathbf{T}_g(\mathbf{x}(\rho)) \\
&\qquad + \sum_{\jmath=1}^{n} \big(x_\jmath(\rho) - x_{\jmath 1g}\big) \\
&\qquad \times \big(x_{\jmath 2g} - x_\jmath(\rho)\big) \mathbf{U}_{\jmath g}(\mathbf{x}(\rho))\Big).
\end{aligned} \tag{30}$$

According to (30), we can achieve $\Delta V(\mathbf{x}(\rho)) < 0$ by

$$\begin{aligned}
&\sum_{d=1}^{s} \sum_{c=1}^{j} \Bigg(\Big(\varphi_{dcq_1q_2\cdots q_ng}(\mathbf{x}(\rho)) + \underline{\delta}_{dcg}\Big) \mathbf{Q}_{dc}(\mathbf{x}(\rho)) + \Big(\overline{\delta}_{dcg} - \underline{\delta}_{dcg}\Big) \\
&\quad \times \mathbf{Y}_{dcg}(\mathbf{x}(\rho)) + \big(\overline{\zeta}_{dcg} - \varphi_{dcq_1q_2\cdots q_ng}(\mathbf{x}(\rho))\big) \mathbf{S}_{dcg}(\mathbf{x}(\rho)) \\
&\quad + \Big(\varphi_{dcq_1q_2\cdots q_ng}(\mathbf{x}(\rho)) - \underline{\zeta}_{dcg}\Big) \mathbf{SS}_{dcg}(\mathbf{x}(\rho)) \\
&\quad + \big(x_\varpi(\rho) - x_{\varpi q_\varpi g}\big) \times \mathbf{T}_g(\mathbf{x}(\rho)) \\
&\quad + \sum_{\jmath=1}^{n} \big(x_\jmath(\rho) - x_{\jmath 1g}\big)\big(x_{\jmath 2g} - x_\jmath(\rho)\big) \mathbf{U}_{\jmath g}(\mathbf{x}(\rho))\Bigg) \prec 0
\end{aligned} \tag{31}$$

guaranteeing the stability of DPF closed-loop system with time delay (6). From (31), together with guaranteed positivity based on Remark 4, the positivity of DPF closed-loop system with time delay (6) is guaranteed if the following theorem is satisfied.

Theorem 1: For the DPF closed-loop system with time delay (6), positivity and asymptotically stability can be guaranteed if there exist $\boldsymbol{\eta} \in \Re^n$ and $\Lambda_w^c(\mathbf{x}(\rho)) \in \Re^m$ for $c \in \underline{j}$, $w \in \underline{n}$ and polynomial vectors $\mathbf{Y}_{dcg}(\mathbf{x}(\rho)) \in \Re^n$, $\mathbf{S}_{dcg}(\mathbf{x}(\rho)) \in \Re^n$, $\mathbf{SS}_{dcg}(\mathbf{x}(\rho)) \in \Re^n$, $\mathbf{T}_g(\mathbf{x}(\rho)) \in \Re^n$, $\mathbf{U}_{\jmath g}(\mathbf{x}(\rho)) \in \Re^n$ and the following SOS-based conditions together with Remark 4 are satisfied:

$$\eta_k - \varepsilon_1 \text{ is SOS}, k \in \underline{n} \tag{32}$$

$$a_{rk}^{d0}(\mathbf{x}(\rho))\eta_k + b_r^d(\mathbf{x}(\rho))\Lambda_k^c(\mathbf{x}(\rho)) \text{ is SOS} \quad d \in \underline{s}, c \in \underline{j}, r, k \in \underline{n} \tag{33}$$

$$a_{rk}^{di} \text{ is SOS}, \ d \in \underline{s}, i \in \underline{l}, r, k \in \underline{n} \tag{34}$$

$$
\begin{aligned}
-\Bigg(\sum_{d=1}^{s}\sum_{c=1}^{j}\Bigg(&\Big(\varphi_{dcq_1q_2\cdots q_ng}(\mathbf{x}(\rho))+\underline{\delta}_{dcg}\Big)q_k^{dc}(\mathbf{x}(\rho))\\
&+\Big(\overline{\delta}_{dcg}-\underline{\delta}_{dcg}\Big)y_k^{dcg}(\mathbf{x}(\rho))\\
&+\big(\overline{\zeta}_{dcg}-\varphi_{dcq_1q_2\cdots q_ng}(\mathbf{x}(\rho))\big)s_k^{dcg}(\mathbf{x}(\rho))\\
&+\Big(\varphi_{dcq_1q_2\cdots q_ng}(\mathbf{x}(\rho))-\underline{\zeta}_{dcg}\Big)ss_k^{dcg}(\mathbf{x}(\rho))\\
&+\big(x_{\varpi}(\rho)-x_{\varpi q_{\varpi}g}\big)t_k^{g}(\mathbf{x}(\rho))\\
&+\sum_{\jmath=1}^{n}\big(x_{\jmath}(\rho)-x_{\jmath 1g}\big)\big(x_{\jmath 2g}-x_{\jmath}(\rho)\big)u_k^{\jmath g}(\mathbf{x}(\rho))\Bigg)\\
&+\ \varepsilon_2(\mathbf{x}(\rho))\Bigg)
\end{aligned}
$$

$$\text{is SOS},\ d\in\underline{s},\ c\in\underline{j},\ g\in\underline{e},\ k,\varpi,\jmath\in\underline{n} \tag{35}$$

$$y_k^{dcg}(\mathbf{x}(\rho))\ \text{is SOS},\ d\in\underline{s},\ c\in\underline{j},\ g\in\underline{e},\ k\in\underline{n} \tag{36}$$

$$y_k^{dcg}(\mathbf{x}(\rho))-q_k^{dc}(\mathbf{x}(\rho))\ \text{is SOS},\ d\in\underline{s},\ c\in\underline{j},\ g\in\underline{e},\ k\in\underline{n} \tag{37}$$

$$s_k^{dcg}(\mathbf{x}(\rho))\ \text{is SOS},\ d\in\underline{s},\ c\in\underline{j},\ g\in\underline{e},\ k\in\underline{n} \tag{38}$$

$$ss_k^{dcg}(\mathbf{x}(\rho))\ \text{is SOS},\ d\in\underline{s},\ c\in\underline{j},\ g\in\underline{e},\ k\in\underline{n} \tag{39}$$

$$u_k^{\jmath g}(\mathbf{x}(\rho))\ \text{is SOS},\ g\in\underline{e},\ k,\jmath\in\underline{n} \tag{40}$$

where matrices $\mathbf{A}_{d0}(\mathbf{x}(\rho))$ and $\mathbf{A}_{di}$ the (r,k)th element are represented as $a_{rk}^{d0}(\mathbf{x}(\rho))$ and a_{rk}^{di}, respectively; vector $\mathbf{B}_d(\mathbf{x}(\rho))$ the rth element is represented as $b_r^d(\mathbf{x}(\rho))$; vectors $\boldsymbol{\eta}$, $\mathbf{Q}_{dc}(\mathbf{x}(\rho))$, $\mathbf{Y}_{dcg}(\mathbf{x}(\rho))\in\Re^n$, $\mathbf{S}_{dcg}(\mathbf{x}(\rho))\in\Re^n$, $\mathbf{SS}_{dcg}(\mathbf{x}(\rho))\in\Re^n$, $\mathbf{T}_g(\mathbf{x}(\rho))\in\Re^n$, and $\mathbf{U}_{\jmath g}(\mathbf{x}(\rho))\in\Re^n$ the kth elements are represented as η_k, $q_k^{dc}(\mathbf{x}(\rho))$, $y_k^{dcg}(\mathbf{x}(\rho))$, $s_k^{dcg}(\mathbf{x}(\rho))$, $ss_k^{dcg}(\mathbf{x}(\rho))$, $t_k^g(\mathbf{x}(\rho))$, and $u_k^{\jmath g}(\mathbf{x}(\rho))$, respectively; scalar satisfies $\varepsilon_1>0$ and predefined scalar polynomial satisfies $\varepsilon_2(\mathbf{x}(\rho))>0$; $q_k^{dc}(\mathbf{x}(\rho))$ is defined in (23).

Proof: The obtained SOS-based stability and positivity conditions (32) to (40) will be achieved based on Remark 4 and Lyapunov stability theory which has been carried out from (20) to (31). ■

Remark 8: As the information of the membership functions is carried by slack matrices to the stability analysis and the number of stability conditions are generally higher, computational demand on finding a feasible solution to the stability conditions will be higher, however, in the membership function dependent stability analysis, the stability conditions obtained are not for any shape of membership functions but dedicated to the fuzzy-model-based control system with time delay to be controlled, which means we can much more easily find the wanted fuzzy controller to stabilize the nonlinear system. Therefore, the conservativeness of stability analysis can be reduced further.

IV. Simulation Example

To validate the proposed Theorem 1 in the previous sections, a simulation example is given here. In the example, we consider a three rule polynomial fuzzy system with time-delay, the system, time delay, and input matrices are given

$$
\begin{aligned}
\mathbf{x}(\rho)&=\begin{bmatrix}x_1(\rho) & x_2(\rho)\end{bmatrix}^T\\
\mathbf{A}_{10}(x_1(\rho))&=\begin{bmatrix}0.02b+0.4+0.015x_1(\rho)-0.001x_1(\rho)^2 & 0.2\\ 0.2 & 0.3\end{bmatrix}\\
\mathbf{A}_{20}(x_1(\rho))&=\begin{bmatrix}0.4 & 0.1-0.01x_1(\rho)\\ 0.2 & 0.01a\end{bmatrix}\\
\mathbf{A}_{30}(x_1(\rho))&=\begin{bmatrix}0.03 & 0.4\\ 0.24+0.01x_1(\rho) & 0.06+0.0003x_1(\rho)^2\end{bmatrix}\\
\mathbf{B}_1(x_1(\rho))&=\begin{bmatrix}0.2b+0.1\\ 0.1-0.001x_1(\rho)^2\end{bmatrix}\\
\mathbf{B}_2(x_1(\rho))&=\begin{bmatrix}1+0.003x_1(\rho)^2\\ 0.1-0.001x_1(\rho)^2\end{bmatrix}\\
\mathbf{B}_3(x_1(\rho))&=\begin{bmatrix}1+0.005x_1(\rho)^2\\ 0.1-0.001x_1(\rho)^2\end{bmatrix}\\
\mathbf{A}_{11}=\mathbf{A}_{21}=\mathbf{A}_{31}&=\begin{bmatrix}0.01 & 0\\ 0 & 0\end{bmatrix}\\
\mathbf{A}_{12}=\mathbf{A}_{22}=\mathbf{A}_{32}&=\begin{bmatrix}0.01 & 0\\ 0 & 0\end{bmatrix}
\end{aligned}
$$

where the predefined constant parameters a and b are chosen gap of 1 and 0.19 for $20\le a\le 27$ and $2.14\le b\le 4.04$, respectively.

The three-rule membership function shapes of the polynomial fuzzy model is given as

$$
\begin{aligned}
w_1(x_1)&=1-\frac{1}{\left(1+e^{-(x_1-6)}\right)}\\
w_2(x_1)&=1-w_1(x_1)-w_3(x_1)\\
w_3(x_1)&=\frac{1}{\left(1+e^{-(x_1-14)}\right)}.
\end{aligned}
$$

Based on the imperfect premise matching design concept, a two-rule fuzzy controllers are chosen to ensure the positivity and stability of the system. The membership function shape of the fuzzy controller as follows:

$$
\begin{aligned}
m_1(x_1)&=e^{-(x_1-10)^2/12}\\
m_2(x_1)&=\mu_{N_1^2}(x_1)=1-m_1(x_1).
\end{aligned}
$$

The membership function shapes of the polynomial fuzzy model and controller are given in Fig. 1.

In this article, PTSMFs red are designed to approximate membership function and bring membership function knowledge into stability and positivity analysis. Since choosing different number sample points and degree, different shape PTSMFs will be obtained which leads to various approximation error. The approximation error of PTSMFs will be considered in stability and positivity analysis. To illustrate the influence of different expansion degrees and gaps of sample points of PTSMFs in feasible regions which satisfy SOS-based positivity and stability conditions, a comparison table is shown (see Table I).

Regarding the cases 1–3 in Table I with the chosen membership functions of the polynomial fuzzy model and controller, corresponding to each subregion δ_g, the regional lower and upper approximation error between PTSMFs $\hat{h}_{21g}(\mathbf{x}(\rho))$ and $h_{21}(\mathbf{x}(\rho))=w_2(x_1)\times m_1(x_1)$ with case 3 can be obtained

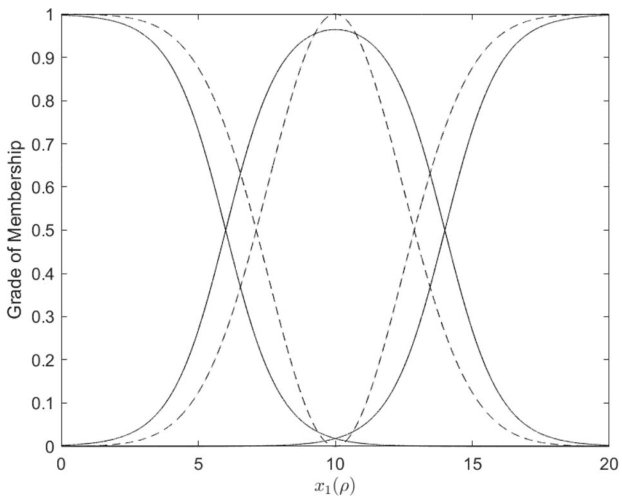

Fig. 1. Solid lines and dotted lines represent the membership functions of the polynomial fuzzy model and polynomial fuzzy controller, respectively.

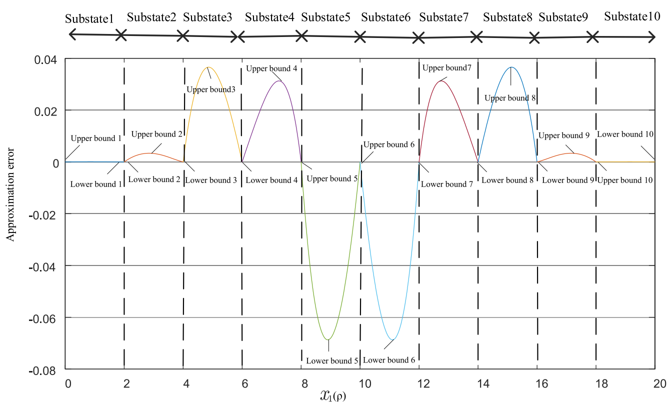

Fig. 2. Regional lower and upper approximation error between PTSMFs $\hat{h}_{21g}(\mathbf{x}(\rho))$ and $h_{21}(\mathbf{x}(\rho)) = w_2(x_1) \times m_1(x_1)$ with case 3 in Table I.

TABLE I
SAMPLE GAPS AND EXPANSION DEGREE OF PTSMFs

Case	Expansion degree	Sample gap	Sample points
1	1	5	$x_1(\rho) = \{0, 5, \ldots, 15, 20\}$
2	1	2	$x_1(\rho) = \{0, 2, \ldots, 18, 20\}$
3	2	2	$x_1(\rho) = \{0, 2, \ldots, 18, 20\}$

as shown in Fig. 2. In the example of case 1, regional lower and upper bound of approximation error between PTSMFs and original membership function $\underline{\delta}_{dcg}$ and $\overline{\delta}_{dcg}$, the regional lower and upper bound of PTSMFs $\underline{\zeta}_{dcg}$ and $\overline{\zeta}_{dcg}$ computed numerically are shown in Table II.

First, polynomial fuzzy controllers of basic stability and positivity condition derived from Remarks 4 and 6 are checked, whose membership function knowledge is considered. $\Lambda_w^c(\mathbf{x}(\rho))$ is set as polynomial of degree 0 to 4 in $x_1(\rho)$ for $c \in \underline{j}, w \in \underline{n}$. And no feasible region is found.

Now, SOS-based positivity and stability conditions which is acquired from Theorem 1 are used to synthesis the feedback gains of polynomial fuzzy controller. In order to ease the computational demand, we reduce the number of slack vectors we used by $\mathbf{Y}_{dcg}(\mathbf{x}(\rho)) = \mathbf{Y}_{dc}(\mathbf{x}(\rho))$, $\mathbf{S}_{dcg}(\mathbf{x}(\rho)) = \mathbf{S}_{dc}(\mathbf{x}(\rho))$, $\mathbf{SS}_{dcg}(\mathbf{x}(\rho)) = \mathbf{SS}_{dc}(\mathbf{x}(\rho))$, $\mathbf{T}_g(\mathbf{x}(\rho)) = \mathbf{T}(\mathbf{x}(\rho))$, and $\mathbf{U}_{Jg}(\mathbf{x}(\rho)) = \mathbf{U}(\mathbf{x}(\rho))$. We set polynomial $\Lambda_w^c(\mathbf{x}(\rho))$ with

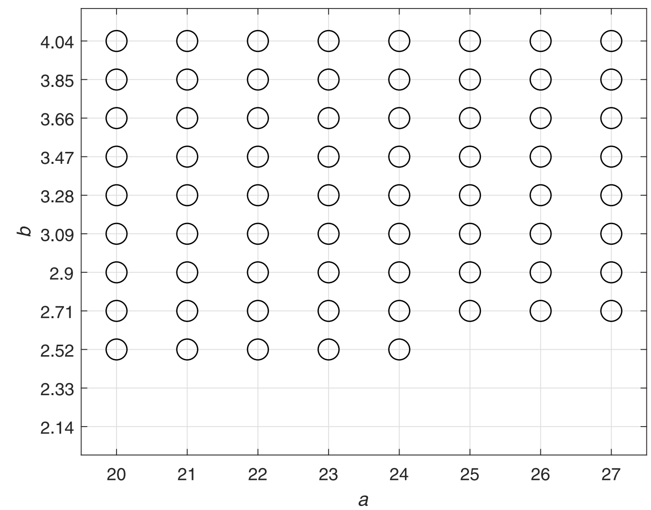

Fig. 3. Feasible regions based on Theorem 1 represented by "×" regarding case 1, "□" regarding case 2, and "∘" regarding case 3 with slack vectors $\mathbf{Y}_{dc}(\mathbf{x}(\rho))$, $\mathbf{S}_{dc}(\mathbf{x}(\rho))$, and $\mathbf{SS}_{dc}(\mathbf{x}(\rho))$.

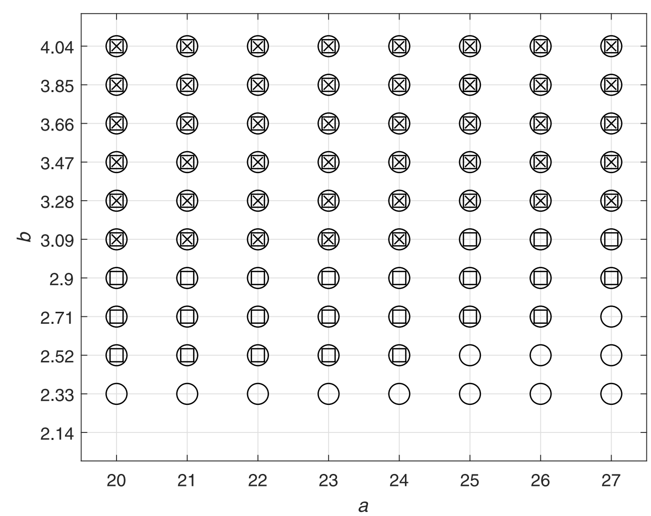

Fig. 4. Feasible regions based on Theorem 1 represented by "×" regarding case 1, "□" regarding case 2, and "∘" regarding case 3 with all slack vectors $\mathbf{Y}_{dc}(\mathbf{x}(\rho))$, $\mathbf{S}_{dc}(\mathbf{x}(\rho))$, $\mathbf{SS}_{dc}(\mathbf{x}(\rho))$, $\mathbf{U}(\mathbf{x}(\rho))$, and $\mathbf{T}(\mathbf{x}(\rho))$.

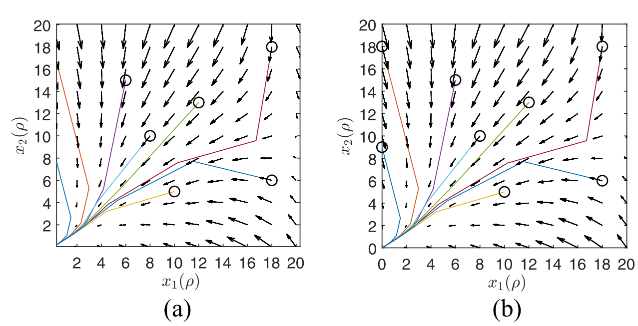

Fig. 5. Referring to feasible regions represented by the mark "∘" regarding case 3 in Fig. 3, phase plots Fig. 5(a) and (b) are chosen parameters $a = 24; b = 2.52$ with time delay $\tau_1 = 50, \tau_2 = 100$ and $\tau_1 = 100, \tau_2 = 200$, respectively.

TABLE II
PARAMETERS $\underline{\delta}_{dcg}$, $\bar{\delta}_{dcg}$, $\underline{\zeta}_{dcg}$, AND $\bar{\zeta}_{dcg}$ FOR CASE 1 REFERRING TABLE I

δ_g	parameters regarding case 1
δ_1	$\underline{\delta}_{111}=-0.0387, \underline{\delta}_{121}=0.0000, \underline{\delta}_{211}=-0.0214,$ $\underline{\delta}_{221}=-0.0957, \underline{\delta}_{311}=-0.0000, \underline{\delta}_{321}=-0.0000,$ $\bar{\delta}_{111}=-0.0000, \bar{\delta}_{121}=0.1542, \bar{\delta}_{211}=0.0000,$ $\bar{\delta}_{221}=-0.0000, \bar{\delta}_{311}=0.0000, \bar{\delta}_{321}=0.0000,$ $\underline{\zeta}_{111}=0.0002, \underline{\zeta}_{121}=0.6400, \underline{\zeta}_{211}=0,$ $\underline{\zeta}_{221}=0.0025, \underline{\zeta}_{311}=0, \underline{\zeta}_{321}=0,$ $\bar{\zeta}_{111}=0.0910, \bar{\zeta}_{121}=0.9973, \bar{\zeta}_{211}=0.0335,$ $\bar{\zeta}_{221}=0.2354, \bar{\zeta}_{311}=0.0000, \bar{\zeta}_{321}=0.0001.$
δ_2	$\underline{\delta}_{111}=-0.0000, \underline{\delta}_{121}=-0.2481, \underline{\delta}_{211}=-0.0921,$ $\underline{\delta}_{221}=-0.0069, \underline{\delta}_{311}=-0.0093, \underline{\delta}_{321}=-0.0000,$ $\bar{\delta}_{111}=0.0677, \bar{\delta}_{121}=0, \bar{\delta}_{211}=0.0924,$ $\bar{\delta}_{221}=0.2455, \bar{\delta}_{311}=0, \bar{\delta}_{321}=0.0007,$ $\underline{\zeta}_{111}=0.0180, \underline{\zeta}_{121}=0, \underline{\zeta}_{211}=0.0335,$ $\underline{\zeta}_{221}=0, \underline{\zeta}_{311}=0.0000, \underline{\zeta}_{321}=0,$ $\bar{\zeta}_{111}=0.0910, \bar{\zeta}_{121}=0.6400, \bar{\zeta}_{211}=0.9640,$ $\bar{\zeta}_{221}=0.2354, \bar{\zeta}_{311}=0.0180, \bar{\zeta}_{321}=0.0001.$
δ_3	$\underline{\delta}_{111}=-0.0093, \underline{\delta}_{121}=-0.0000, \underline{\delta}_{211}=-0.0921,$ $\underline{\delta}_{221}=-0.0069, \underline{\delta}_{311}=-0.0000, \underline{\delta}_{321}=-0.2481,$ $\bar{\delta}_{111}=0.0000, \bar{\delta}_{121}=0.0007, \bar{\delta}_{211}=0.0924,$ $\bar{\delta}_{221}=0.2455, \bar{\delta}_{311}=0.0677, \bar{\delta}_{321}=0,$ $\underline{\zeta}_{111}=0.0000, \underline{\zeta}_{121}=0, \underline{\zeta}_{211}=0.0335,$ $\underline{\zeta}_{221}=0, \underline{\zeta}_{311}=0.0180, \underline{\zeta}_{321}=0,$ $\bar{\zeta}_{111}=0.0180, \bar{\zeta}_{121}=0.0001, \bar{\zeta}_{211}=0.9640,$ $\bar{\zeta}_{221}=0.2354, \bar{\zeta}_{311}=0.0910, \bar{\zeta}_{321}=0.6400.$
δ_4	$\underline{\delta}_{111}=-0.0000, \underline{\delta}_{121}=-0.0000, \underline{\delta}_{211}=-0.0214,$ $\underline{\delta}_{221}=-0.0957, \underline{\delta}_{311}=-0.0387, \underline{\delta}_{321}=0,$ $\bar{\delta}_{111}=0.0000, \bar{\delta}_{121}=0.0000, \bar{\delta}_{211}=0.0000,$ $\bar{\delta}_{221}=-0.0000, \bar{\delta}_{311}=-0.0000, \bar{\delta}_{321}=0.1542,$ $\underline{\zeta}_{111}=0, \underline{\zeta}_{121}=0, \underline{\zeta}_{211}=0,$ $\underline{\zeta}_{221}=0.0025, \underline{\zeta}_{311}=0.0002, \underline{\zeta}_{321}=0.6400,$ $\bar{\zeta}_{111}=0.0000, \bar{\zeta}_{121}=0.0001, \bar{\zeta}_{211}=0.0335,$ $\bar{\zeta}_{221}=0.2354, \bar{\zeta}_{311}=0.0910, \bar{\zeta}_{321}=0.9973.$

degree 0 to 4 in $x_1(\rho)$ for $c \in \underline{j}, w \in \underline{n}$ and $\varepsilon_1 = \varepsilon_2(\mathbf{x}(\rho)) = 0.0010$ for solver initialization.

We first employ Theorem 1 only with slack vectors $\mathbf{Y}_{dc}(\mathbf{x}(\rho))$, $\mathbf{S}_{dc}(\mathbf{x}(\rho))$, and $\mathbf{SS}_{dc}(\mathbf{x}(\rho))$ all as polynomial of degree 0 in $x_1(\rho)$ to synthesize polynomial fuzzy controller. This means only regional approximation error between PTSMFs and original membership function with the regional bounds of PTSMFs are considered in the stability analysis. Fig. 3 provides the resulting feasible regions.

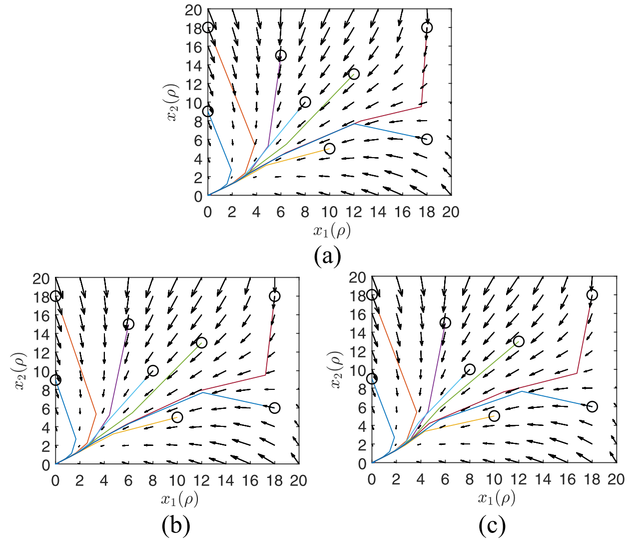

Fig. 6. Referring to feasible regions in Fig. 4 represented by the mark "×" regarding case 1, phase plot Fig. 6(a) are chosen parameters $a = 24$; $b = 3.09$; represented by the mark "□" regarding case 2, phase plot Fig. 6(b) are chosen parameters $a = 24$; $b = 2.52$; represented by the mark "∘" regarding case 3, phase plot Fig. 6(c) are chosen parameters $a = 27$; $b = 2.33$. All with time delay $\tau_1 = 50$, $\tau_2 = 100$.

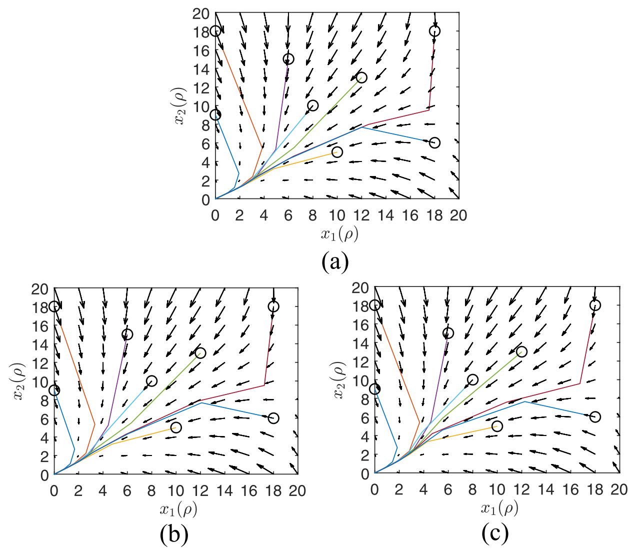

Fig. 7. Referring to feasible regions in Fig. 4 represented by the mark "×" regarding case 1, phase plot Fig. 7(a) are chosen parameters $a = 24$; $b = 3.09$; represented by the mark "□" regarding case 2, phase plot Fig. 7(b) are chosen parameters $a = 24$; $b = 2.52$; represented by the mark "∘" regarding case 3, phase plot Fig. 7(c) are chosen parameters $a = 27$; $b = 2.33$. All with time delay $\tau_1 = 100$, $\tau_2 = 200$.

In the next step, we employ Theorem 1 with all slack vectors to synthesis polynomial fuzzy controller. We set slack vectors $\mathbf{Y}_{dc}(\mathbf{x}(\rho))$, $\mathbf{U}(\mathbf{x}(\rho))$, $\mathbf{S}_{dc}(\mathbf{x}(\rho))$, and $\mathbf{SS}_{dc}(\mathbf{x}(\rho))$ all as polynomial of degree 0 in $x_1(\rho)$ and $\mathbf{T}(\mathbf{x}(\rho))$ as polynomial of degree 0 to 2 in $x_1(\rho)$ for $d \in \underline{s}, c \in \underline{j}$. This means the regional bounds of PTSMFs, interpolation membership function the property knowledge of PTSMFs and regional bounds of premise variables together with regional approximated error between PTSMFs and original membership function are considered in the stability and positivity analysis. The resulting feasible regions are shown in Fig. 4.

TABLE III
POLYNOMIAL FUZZY CONTROLLER OBTAINED IN FIGS. 3 AND 4

Case	Theorem 1 for Fig. 3	
	a, b	Polynomial fuzzy controller
3	24, 2.52	$\mathbf{G}_1(x_1) = [0.1553 \times 10^{-7}x_1^4 - 0.3458 \times 10^{-5}x_1^3 + 0.0013x_1^2 - 0.0117x_1 + 0.0174, -0.1376 \times 10^{-7}x_1^4 - 0.1525 \times 10^{-5}x_1^3 -0.9689 \times 10^{-4}x_1^2 + 0.0054x_1 - 0.0403]$
		$\mathbf{G}_2(x_1) = [0.1881 \times 10^{-7}x_1^4 - 0.3238 \times 10^{-5}x_1^3 + 0.0013x_1^2 - 0.0135x_1 + 0.0152, -0.1539 \times 10^{-7}x_1^4 - 0.1529 \times 10^{-5}x_1^3 -0.7584 \times 10^{-4}x_1^2 + 0.0060x_1 - 0.0580]$
Case	Theorem 1 for Fig. 4	
	a, b	Polynomial fuzzy controller
1	24, 3.09	$\mathbf{G}_1(x_1) = [0.4087 \times 10^{-7}x_1^4 - 0.3248 \times 10^{-5}x_1^3 + 0.0011x_1^2 - 0.0108x_1 + 0.0450, -0.1922 \times 10^{-7}x_1^4 + 0.1579 \times 10^{-6}x_1^3 +0.3575 \times 10^{-4}x_1^2 + 0.0008x_1 + 0.0177]$
		$\mathbf{G}_2(x_1) = [0.4283 \times 10^{-7}x_1^4 - 0.3098 \times 10^{-5}x_1^3 + 0.0010x_1^2 - 0.0112x_1 + 0.0437, -0.1329 \times 10^{-7}x_1^4 - 0.2135 \times 10^{-6}x_1^3 +0.8774 \times 10^{-6}x_1^2 + 0.0004x_1 + 0.0180]$
2	24, 2.52	$\mathbf{G}_1(x_1) = [0.2976 \times 10^{-7}x_1^4 - 0.3360 \times 10^{-5}x_1^3 + 0.0013x_1^2 - 0.0115x_1 + 0.0252, -0.1120 \times 10^{-7}x_1^4 - 0.2108 \times 10^{-6}x_1^3 -0.1195 \times 10^{-5}x_1^2 + 0.0038x_1 - 0.0195]$
		$\mathbf{G}_2(x_1) = [0.4491 \times 10^{-7}x_1^4 - 0.2608 \times 10^{-5}x_1^3 + 0.0013x_1^2 - 0.0145x_1 + 0.0315, -0.3457 \times 10^{-10}x_1^4 + 0.8376 \times 10^{-7}x_1^3 -0.4297 \times 10^{-4}x_1^2 + 0.0029x_1 - 0.0255]$
3	27, 2.33	$\mathbf{G}_1(x_1) = [0.3211 \times 10^{-7}x_1^4 - 0.4226 \times 10^{-5}x_1^3 + 0.1334 \times 10^{-2}x_1^2 - 0.0156x_1 +0.0490, -0.3430 \times 10^{-7}x_1^4 - 0.3469 \times 10^{-6}x_1^3 - 0.6757 \times 10^{-4}x_1^2 +0.8669 \times 10^{-4}x_1 + 0.0074]$
		$\mathbf{G}_2(x_1) = [0.3296 \times 10^{-7}x_1^4 - 0.5275 \times 10^{-5}x_1^3 + 0.1406 \times 10^{-2}x_1^2 - 0.0151x_1 +0.0435, -0.3516 \times 10^{-7}x_1^4 - 0.1740 \times 10^{-5}x_1^3 + 0.2437 \times 10^{-4}x_1^2 -0.2753 \times 10^{-3}x_1 + 0.0048]$

As shown in Figs. 3 and 4, it verifies that the basic membership independent stability and positivity analysis derived from Remarks 4 and 6 is very conservative since no feasible regions founded. The stability and positivity conditions derived from Theorem 1 can be relaxed by introducing the approximated error between PTSMFs and original membership functions into the stability and positivity analysis. Furthermore, it can be found that PTSMFs possessing the higher expansion degree and smaller gap sample points for case 3 find largest feasible regions, which indicates case 3 can provide smaller approximated error compared with cases 1 and 2 and introduce more membership function knowledge into stability and positivity analysis which leads to more feasible regions. It is also evident that Theorem 1 (with all slack vectors) in Fig. 4 offers the larger feasible regions when compared with Theorem 1 [only with slack vectors $\mathbf{Y}_{dc}(\mathbf{x}(\rho))$, $\mathbf{S}_{dc}(\mathbf{x}(\rho))$, and $\mathbf{SS}_{dc}(\mathbf{x}(\rho))$] in Fig. 3. This means more information from membership function and premise variables carried by corresponding slack vectors can effectively reduce the conservativeness of stability and positivity conditions.

To verify, DPF closed-loop system states are tested through the phase plots corresponding to each situation in Figs. 3 and 4. To achieve the best results, the trajectory of $x_1(\rho)$ and $x_2(\rho)$ are simulated through eight different initial conditions indicated with "$\circ$." As noticed from Figs. 5 to 7, the system states are guaranteed positive while the DPF controller driving the system state to equilibrium (origin) starting from any initial condition. Furthermore, the system positivity and stability time delay independent as we change $\tau_1 = 50$ to $\tau_2 = 100$ [see Figs. 5(a) and 6], and $\tau_1 = 100$ to $\tau_2 = 200$ [see Figs. 5(b) and 7]. This is because the positivity and stability conditions described in Theorem 1 are essentially independent of the delay period. The corresponding polynomial fuzzy controller shown in Table III.

V. CONCLUSION

This article investigates the positivity and stability of DPF closed-loop system with time delay. To reduce the conservativeness of stability and positivity analysis, PTSMFs are proposed to approximate the original membership function. We showed that the introduction of regional approximation error can relax the derived stability and positivity conditions. To further relax the conditions, the regional bounds of PTSMFs, the property knowledge of interpolation membership function of PTSMFs and bounds of premise variables together with PTSMFs are imposed on the stability and positivity analysis. We validated our proposed Theorem 1 and the formulated SOS-based stability and positivity conditions therein via the given simulation example.

In terms of future research direction, the stability and positivity conditions of discrete time PFMB control system with time delay by combining other control methods, such as output-feedback and observer-based feedback controller. Furthermore, both widely application in communication systems and formation flying and the theoretical challenge of switched positive systems with time delays, there will be a big motivation to study such different kinds of system.

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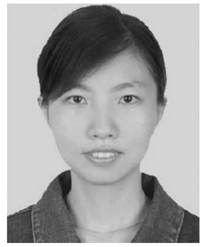

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