

Model-Based Control and Stability Analysis of Discrete-Time Polynomial Fuzzy Systems With Time Delay and Positivity Constraints

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Abstract—This paper proposes a novel Lyapunov stabilization analysis of discrete-time polynomial-fuzzy-model-based control systems with time delay under positivity constraint. The polynomial fuzzy model is constructed to describe the dynamics of a nonlinear discrete-time system with time delay. A model-based polynomial fuzzy controller is designed using nonparallel distributed compensation technique to stabilize the system while driving the system states to positive using the positivity constraints. The Lyapunov stability and positivity conditions are formulated as sum-of-squares. To relax the conservativeness of the obtained stability results, two main methods are proposed in this paper: first, the piecewise linear membership functions (PLMFs) are used to introduce the approximate error between piecewise and the original membership functions into the stability analysis; and second, introduce the boundary information of the premise variables into the stability analysis since the premise variables hold rich nonlinear information. A numerical example is given to demonstrate the effectiveness of the proposed approach.

Index Terms—Discrete-time polynomial fuzzy-model-based (PFMB) control systems with time delay, positive systems, piecewise linear membership functions (PLMFs), premise variables, stability analysis, SOS.

I. INTRODUCTION

A SPECIAL class of physical systems in the real world involve quantities that have an intrinsically constant sign, such as absolute temperatures, the concentration of substances in the chemical process, level of liquids in tanks, and so on [1]–[5]. This means their states are always nonnegative if the initial conditions are nonnegative. There can be found real-life examples of positive systems in biology, economics, etc., and a noticeable amount of work has been done to control and stabilize the positive systems [1]–[10].

The response of real-world systems to external signals is always associated with time delays, which can trigger complex behavior and result in poor performance and instability. The

stability of positive discrete-time systems with time delay has been an increasingly important topic of study [11]–[13]. Powerful stability analysis methods such as traditional Lyapunov–Krasovskii functional method [14], [15] can be applied to positive discrete-time systems with a time delay as a special case of discrete-time systems with time delay. The disadvantage of this general method is that it neglects the innate features of positivity, leading to conservativeness stability conditions. Therefore, linear copositive function is proposed and widely employed to consider the innate features of positivity for the stability and positivity analyses of positive systems [16]–[19]. The stability conditions based on linear copositive function is independent of the time delay and its value. Using universal approximation capability, T–S fuzzy model is able to accurately represent nonlinear positive discrete-time systems with time delay [16], [19]–[21]. Moreover, the technique of weighted sum of certain local linear subsystems can provide a useful tool for the stability analysis and control synthesis of this type of systems [16], [19], [21]–[26]. The traditional parallel distributed compensation (PDC) technique can also be used to synthesize the fuzzy controller based on discrete-time T–S fuzzy-model with time delay [16], [19], [24]–[26], and to formulate the stability and positivity conditions as linear matrix inequalities (LMIs) [16], [22], [27].

Polynomial fuzzy models have been proposed to improve the approximation capability of a T–S fuzzy model and represent nonlinear positive continuous systems with time delay [28]–[30]. This modeling approach essentially replaces the T–S local linear subsystems by polynomial ones. The feedback gains of fuzzy polynomial controllers are then defined in the polynomial form. The polynomial representation enhances the modeling and feedback compensation [29]–[38] since the nonlinear system is represented more accurately due to the introduction of nonlinear polynomial terms, and the stability analysis becomes more inclusive as the polynomial fuzzy model is a global model when compared to the original T–S fuzzy model. Moreover, the nonlinear polynomial terms are no longer necessary to be represented using membership functions resulting in the reduction of the number of fuzzy rules. The feasible solutions to a polynomial Lyapunov stability and positivity formulation can be found using third party MATLAB toolbox SOSTOOLS [39], [40] where LMI fails. The advantages of polynomial fuzzy models and lack of existing literature on representing nonlinear positive discrete-time system with time delay using polynomial

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models, motivates us to represent the dynamics of nonlinear positive discrete-time system with time delay with polynomial fuzzy model and develop the stability analysis and control synthesis based on discrete-time PFMB control systems with time delay.

After the original paper of Tanaka *et al.* [41], the PDC technique has been used extensively as an effective technique to design fuzzy model-based controllers. In the PDC design concept, the conservativeness of stability conditions can be relaxed to some extent by considering the matched premise membership functions. However, this technique requires the number of rules and shape of membership functions of a fuzzy model to match with those of a fuzzy controller. This constraint limits the design flexibility of a fuzzy controller. Non-PDC technique is proposed to remove such constraint when designing fuzzy rules for the controller [29]–[31]. Membership functions and premise variables play important roles for relaxing the stability conditions [31]–[34], [42], [43]. To relax the stability conditions for positive continuous fuzzy-model-based control systems with time delay, a few attempts have been made to introduce the information of membership functions in the stability analysis [29], [30]. Both methods achieved the relaxed conditions by introducing the information of membership function in stability analysis using PLMFs [29] or symbolic variables [30]. However, there is no existing literature on extending the novel methods to positive *discrete-time* fuzzy-model-based control systems with time delay. The premise variables, besides membership functions, contain information of the system nonlinearity which can be exerted to relax the stability conditions if introduced in the stability formulations. Therefore, there is a strong motivation for this body of work to use the information of premise variables for discrete-time positive fuzzy-model-based control systems with time delay.

The contribution of this paper can be listed as follows.

- 1) Representing positive discrete-time systems with time delay using a polynomial fuzzy model.
- 2) Non-PDC design concept is used to design the controller for polynomial fuzzy model-based systems with time delay, and investigate the stability and positivity analysis, leading to an increase to the design flexibility by choosing the number of rules and shape of membership functions of the fuzzy controller freely.
- 3) As the nonlinearity information is embedded in the membership functions and premise variables, piecewise linear membership functions are used to approximate original membership functions, and the information of membership functions, i.e., approximation error between the approximate (piecewise linear) membership functions and the original membership functions is introduced to the stability analysis. In this way, the conservativeness of the stability conditions will be relaxed. The traditional linear copositive function [16]–[19] ignores the membership functions in the derivation process of the Lyapunov function candidate, leading to inability in introducing the approximation error. The Lyapunov function candidate is updated where the order of fuzzy model number rules for delay matrix of the later part is designed, independent of

the order of fuzzy model number rules for the system and delay matrix of the previous part. This allows us to introduce the information of membership functions further in the process.

- 4) To introduce more nonlinear information into the stability analysis, we consider the boundary information of the premise variables in addition to the information of piecewise linear membership functions to further relax the stability conditions.

The rest of the paper is organized as follows. In section II, the details of discrete-time polynomial fuzzy model and controller with time delay are described. In Section III, the SOS Lyapunov stability and positivity conditions are derived based on the non-PDC concept. In Section IV, a numerical example is discussed to show the validity of the proposed methods, and finally in Section V, a conclusion will be drawn.

II. NOTATIONS AND PRELIMINARIES

A. Notation

Throughout this paper, the following notations are adopted. A monomial in $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]$ is a function of the form $x_1^{d_1}(t), x_2^{d_2}(t), \dots, x_n^{d_n}(t)$, where $d_i, i \in \{1, 2, \dots, n\}$ is a nonnegative integer. The degree of a monomial is defined as $d = \sum_{i=1}^n d_i$. $\mathbf{p}(\mathbf{x}(t))$, which is a polynomial if expressed as a finite linear combination of monomials with real coefficients. The equation $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^m \mathbf{q}_j(\mathbf{x}(t))^2$ is an SOS fulfilling the condition $\mathbf{p}(\mathbf{x}(t)) \geq 0$, where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a nonzero positive integer.¹

B. Discrete-Time Polynomial Fuzzy Models With Time Delay

The dynamics of nonlinear discrete-time systems with time delay is represented by p fuzzy rules of discrete-time polynomial fuzzy systems with time delay, in which the i th rule is presented as follows:

$$\begin{aligned} \text{Rule } i: \quad & \text{IF } f_1(\mathbf{x}(k)) \text{ is } M_1^i \text{ AND } \dots \text{ AND } f_\Psi(\mathbf{x}(k)) \text{ is } M_\Psi^i \\ & \text{THEN } \mathbf{x}(k+1) = \mathbf{A}_{i0}(\mathbf{x}(k))\mathbf{x}(k) \\ & \quad + \sum_{l=1}^d \mathbf{A}_{il}\mathbf{x}(k-\tau_l) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{u}(k) \quad (1) \end{aligned}$$

$$\mathbf{x}(k) = \phi(k), \quad k = [-\tau_{\max}, 0] \quad (2)$$

where $M_l^i, l = \{1, 2, \dots, \Psi\}$ is the fuzzy set of i corresponding to premise variable $f_l(\mathbf{x}(k))$, with $l = \{1, 2, \dots, \Psi\}$; Ψ is a positive integer; $\phi(k)$ is the vector valued initial function; $\mathbf{A}_{i0}(\mathbf{x}(k)) \in \mathbb{R}^{n \times n}$, $\mathbf{A}_{il} \in \mathbb{R}^{n \times n}$ and $\mathbf{B}_i(\mathbf{x}(k)) \in \mathbb{R}^{n \times m}$ are polynomial system, time delay, and input matrices, respectively, with $i \in p = \{1, 2, \dots, p\}$, p is the number of IF-THEN rules; $\mathbf{x}(k) \in \mathbb{R}^n$ and $\mathbf{u}(k) \in \mathbb{R}^m$ are state vector and control input vector, respectively; $\tau_l, l \in \underline{d} = \{1, 2, \dots, d\}$ is the constant time delay and $\tau_{\max} = \max(\tau_l)$.

¹ $\mathbf{Z} \succ 0$, $\mathbf{Z} \succcurlyeq 0$, $\mathbf{Z} \prec 0$, and $\mathbf{Z} \preceq 0$ mean that all the elements of the matrix \mathbf{Z} are positive, semipositive, negative, and seminegative, respectively.

The dynamics of nonlinear systems is given as follows:

$$\mathbf{x}(k+1) = \sum_{i=1}^p w_i(\mathbf{x}(k)) \left(\mathbf{A}_{i0}(\mathbf{x}(k))\mathbf{x}(k) + \sum_{l=1}^d \mathbf{A}_{il}\mathbf{x}(k-\tau_l) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{u}(k) \right) \quad (3)$$

where

$$\sum_{i=1}^p w_i(\mathbf{x}(k)) = 1 \quad (4)$$

$$w_i(\mathbf{x}(k)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(k)))}{\sum_{n=1}^p \prod_{l=1}^{\Psi} \mu_{M_l^n}(f_l(\mathbf{x}(k)))} \quad i \in \underline{p} \quad (5)$$

$$w_i(\mathbf{x}(k)) \geq 0 \quad i \in \underline{p} \quad (6)$$

where $w_i(\mathbf{x}(k))$ is the normalized grade of membership and $\mu_{M_l^i}(f_l(\mathbf{x}(k)))$ is the grade of membership corresponding to the fuzzy term M_l^i .

Definition 1 ([16], [19]): System (3) is said to be positive if the initial condition $\phi(\cdot) \succcurlyeq 0$ holds and the corresponding trajectory $\mathbf{x}(k) \succcurlyeq 0$ for all k .

Lemma 1 ([16], [19]): System (3) is said to be positive if the system and time delay matrices satisfy the conditions that $\mathbf{A}_{i0}(\mathbf{x}(k)) \succcurlyeq 0$ and $\mathbf{A}_{il} \succcurlyeq 0$ when $\mathbf{u}(k) = \mathbf{0}$.

Assumption 1: $\mathbf{A}_{il} \succcurlyeq 0$ is satisfied by the polynomial fuzzy model (3), otherwise, the system is not a positive system or cannot be controlled positive.

C. Discrete-Time Polynomial Fuzzy Controller

Under non-PDC design concept, the j th rule of the polynomial fuzzy controller is given as follows:

$$\begin{aligned} \text{Rule } j: \quad & \text{IF } g_1(\mathbf{x}(k)) \text{ is } N_1^j \text{ AND } \cdots \text{ AND } g_{\Omega}(\mathbf{x}(k)) \text{ is } N_{\Omega}^j \\ & \text{THEN } \mathbf{u}(k) = \mathbf{G}_j(\mathbf{x}(k))\mathbf{x}(k) \end{aligned} \quad (7)$$

where N_l^j is the fuzzy set of j corresponding to the premise variable $g_l(\mathbf{x}(k))$, with $l \in \{1, 2, \dots, \Omega\}$, and Ω is a positive integer; $\mathbf{G}_j(\mathbf{x}(k)) \in \mathbb{R}^{m \times n}$ is the polynomial feedback gain with $j \in \underline{c} = \{1, 2, \dots, c\}$, c is the number of IF-THEN rules. Therefore, the discrete-time polynomial fuzzy controller is given by

$$\mathbf{u}(k) = \sum_{j=1}^c m_j(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))\mathbf{x}(k) \quad (8)$$

where

$$\sum_{j=1}^c m_j(\mathbf{x}(k)) = 1 \quad (9)$$

$$m_j(\mathbf{x}(k)) = \frac{\prod_{l=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(k)))}{\sum_{n=1}^c \prod_{l=1}^{\Omega} \mu_{N_l^n}(g_l(\mathbf{x}(k)))} \quad j \in \underline{c} \quad (10)$$

$$m_j(\mathbf{x}(k)) \geq 0 \quad j \in \underline{c} \quad (11)$$

$m_j(\mathbf{x}(k))$ is the normalized grade of membership and $\mu_{N_l^j}(g_l(\mathbf{x}(k)))$ is the grade of membership corresponding to the fuzzy term N_l^j .

Remark 1 ([44], [45]): Under non-PDC design concept, the membership function $m_j(\mathbf{x}(k))$ of the polynomial fuzzy controller can be chosen $m_j(\mathbf{x}(k)) \neq w_i(\mathbf{x}(k))$ for any j allowed.

III. STABILITY AND POSITIVITY ANALYSES

In this section, the Lyapunov stability and positivity conditions for discrete-time polynomial fuzzy systems with time delay are formulated. First, the formulation of closed-loop discrete-time PFMB control systems with time delay is presented. Then by considering the information of membership functions and premise variables in stability analysis, SOS-based conditions are obtained to guarantee the system stability as well as determine the discrete-time fuzzy controller gains. The closed-loop discrete-time PFMB control system with time delay is described based on the model (3) and the controller (8)

$$\begin{aligned} \mathbf{x}(k+1) = & \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))((\mathbf{A}_{i0}(\mathbf{x}(k))) \\ & + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))\mathbf{x}(k) + \sum_{l=1}^d \mathbf{A}_{il}\mathbf{x}(k-\tau_l)). \end{aligned} \quad (12)$$

Lemma 2 ([16], [19]): The control system (12) is said to be controlled positive if $\mathbf{A}_{i0}(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)) \succcurlyeq 0$ and $\mathbf{A}_{il} \succcurlyeq 0$.

Remark 2: We transfer system (12) to a system called *dual system* [19]. Such transfer is valid as it results in equivalent stability terms. If we keep the original system (12), we cannot obtain a unit expression and eliminate the time delay elements using the Lyapunov stability analysis. Therefore, we need the transfer under duality to ease the stability analysis.

The dual system of (12) is described as follows:

$$\begin{aligned} \mathbf{x}(k+1) = & \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))((\mathbf{A}_{i0}(\mathbf{x}(k))) \\ & + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))^T \mathbf{x}(k) + \sum_{l=1}^d \mathbf{A}_{il}^T \mathbf{x}(k-\tau_l)). \end{aligned} \quad (13)$$

A. Positivity Analysis

Prior to stability analysis, the first control objective is to make the closed loop discrete-time PFMB control systems with time delay positive, i.e., trajectory $\mathbf{x}(k) \succcurlyeq 0$ if the initial condition $\phi(\cdot) \succcurlyeq 0$. The positivity conditions (cf., Lemma 2) are formulated by the following theorem.

Theorem 1: The discrete-time PFMB control system with time delay (12) or dual system (13) with the initial condition $\phi(\cdot) \succcurlyeq 0$ is guaranteed to be positive if there exist $\lambda \in \mathbb{R}^n$ and $\mathbf{y}_k^j(\mathbf{x}(k)) \in \mathbb{R}^m$ for $j \in \underline{c}$ and $k \in \underline{n}$ such that the following

SOS-based conditions are satisfied:

$$a_{rs}^{i0}(\mathbf{x}(k))\lambda_s + \mathbf{b}_r^i(\mathbf{x}(k))\mathbf{y}_s^j(\mathbf{x}(k)) \text{ is SOS,} \\ i \in \underline{p}; j \in \underline{c}; r \neq s \quad (14)$$

$$a_{rs}^{il} \text{ is SOS, } i \in \underline{p}; r \neq s; l \in \underline{d} \quad (15)$$

where $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]^T \succ 0$, $a_{rs}^{i0}(\mathbf{x}(k))$ and a_{rs}^{il} are the (r, s) th element of the matrices $\mathbf{A}_{i0}(\mathbf{x}(k))$ and \mathbf{A}_{il} , respectively; $\mathbf{B}_i(\mathbf{x}(k)) = [\mathbf{b}_1^i(\mathbf{x}(k))^T, \mathbf{b}_2^i(\mathbf{x}(k))^T, \dots, \mathbf{b}_n^i(\mathbf{x}(k))^T]^T$, $i \in \underline{p}$, $r, s \in \underline{n}$; $\mathbf{G}_j(\mathbf{x}(k)) = [\frac{\mathbf{y}_1^j(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{y}_2^j(\mathbf{x}(k))}{\lambda_2}, \dots, \frac{\mathbf{y}_n^j(\mathbf{x}(k))}{\lambda_n}]$ where $\mathbf{y}_1^j(\mathbf{x}(k)), \mathbf{y}_2^j(\mathbf{x}(k)), \dots, \mathbf{y}_n^j(\mathbf{x}(k)) \in \mathbb{R}^m$ for $j \in \underline{c}$ are to be determined.

B. Basic Stability Analysis

The second control objective is to make the closed-loop PFMB control systems with time delay asymptotically stable, i.e., trajectory $\mathbf{x}(k) \rightarrow 0$ as $k \rightarrow \infty$.

Based on Assumption 1, the choice of polynomial Lyapunov function used to investigate the stability of (12) is given by

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k)\lambda + \sum_{m=1}^p \sum_{l=1}^d \sum_{q=1}^{\tau_d} \left(\mathbf{x}(k-q)^T \mathbf{A}_{ml} \lambda \right). \quad (16)$$

From (16) and (13), we have

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) \left[\mathbf{x}^T(k)(\mathbf{A}_{i0}(\mathbf{x}(k)) \right. \\ &\quad \left. + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))) + \mathbf{x}^T(k-\tau_l) \sum_{l=1}^d \mathbf{A}_{il} \right] \lambda \\ &\quad + \sum_{m=1}^p \sum_{l=1}^d \left(\mathbf{x}(k)^T \mathbf{A}_{ml} - \mathbf{x}(k-\tau_l)^T \mathbf{A}_{ml} \right) \lambda \\ &\quad - \mathbf{x}^T(k)\lambda \\ &\leq \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) \left[\mathbf{x}^T(k)(\mathbf{A}_{i0}(\mathbf{x}(k)) \right. \\ &\quad \left. + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))) \right] \lambda \\ &\quad + \sum_{i=1}^p \sum_{l=1}^d \mathbf{x}^T(k-\tau_l) \mathbf{A}_{il} \lambda - \mathbf{x}^T(k)\lambda \\ &\quad + \sum_{m=1}^p \sum_{l=1}^d \left(\mathbf{x}(k)^T \mathbf{A}_{ml} - \mathbf{x}(k-\tau_l)^T \mathbf{A}_{ml} \right) \lambda \\ &\leq \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) \left[\mathbf{x}^T(k)(\mathbf{A}_{i0}(\mathbf{x}(k)) \right. \\ &\quad \left. + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))) \right] \lambda - \mathbf{x}^T(k)\lambda \\ &\quad + \sum_{m=1}^p \sum_{l=1}^d \mathbf{x}(k)^T \mathbf{A}_{ml} \lambda \end{aligned}$$

$$\begin{aligned} &= \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\mathbf{x}^T(k) \left[\left(\mathbf{A}_{i0}(\mathbf{x}(k)) \right. \right. \\ &\quad \left. \left. + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)) \right) \lambda - \lambda + \sum_{m=1}^p \sum_{l=1}^d \mathbf{A}_{ml} \lambda \right] \\ &= \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\mathbf{x}^T(k) \left[\left(\mathbf{A}_{i0}(\mathbf{x}(k)) \right. \right. \\ &\quad \left. \left. + \sum_{m=1}^p \sum_{l=1}^d \mathbf{A}_{ml} \right) \lambda + \mathbf{B}_i(\mathbf{x}(k)) \sum_{s=1}^n \mathbf{y}_s^j(\mathbf{x}(k)) - \lambda \right] \\ &= \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\mathbf{x}^T(k) \mathbf{Q}_{ij}(\mathbf{x}(k)) \quad (17) \end{aligned}$$

where

$$\begin{aligned} \mathbf{Q}_{ij}(\mathbf{x}(k)) &= \left(\mathbf{A}_{i0}(\mathbf{x}(k)) + \sum_{m=1}^p \sum_{l=1}^d \mathbf{A}_{ml} \right) \lambda \\ &\quad + \mathbf{B}_i(\mathbf{x}(k)) \sum_{s=1}^n \mathbf{y}_s^j(\mathbf{x}(k)) - \lambda. \quad (18) \end{aligned}$$

Remark 3: Since the stability analysis of (12) and its dual system (13) are equivalent, the PFMB control system with time delay (12) is guaranteed to be asymptotically stable, if the fuzzy polynomial controller satisfies the conditions $V(k) > 0$ and $\Delta V(\mathbf{x}(k)) < 0$ (excluding $\mathbf{x}(k) = 0$) according to basic Lyapunov stability conditions. This can be achieved by $\mathbf{Q}_{ij}(\mathbf{x}(k)) \prec 0$ for all i and j . The resulting stability conditions however are very conservative as the information of membership functions and premise variables is not considered.

Remark 4: The control system (12) is guaranteed to be asymptotically positive and stable if the stability and positive conditions in Theorem 1 and Remark 3 are satisfied.

C. Membership Functions Dependent Stability Analysis

In the next step, to relax the stability analysis, we use a favorable form called *piecewise linear membership functions* to approximate the original membership function and introduce the information of membership functions into the stability analysis. The concept of piecewise linear membership functions is shown in Fig. 1. The original membership function $f(\mathbf{x}(k))$ is distributed in overall system state space φ . A certain number of sample points based on our requirement are used to divide the state space with D connected subspaces. The l th subspace is denoted as φ_l , $l \in \underline{D} = \{1, 2, \dots, D\}$. The piecewise linear membership function $f_{\varphi_l}(\mathbf{x}(k))$ of every subspace φ_l is obtained based on the linear relationship with boundary sample points of φ_l (see Fig. 1). The shape of piecewise linear membership functions depend on the corresponding subspaces as shown in Fig. 1.

Let us define $h_{ij}(\mathbf{x}(k)) = w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))$, and $\hat{h}_{ij}(\mathbf{x}(k))$ denotes as the piecewise linear membership functions to approximate $h_{ij}(\mathbf{x}(k))$. Hence, the expression of $\hat{h}_{ij}(\mathbf{x}(k))$ is

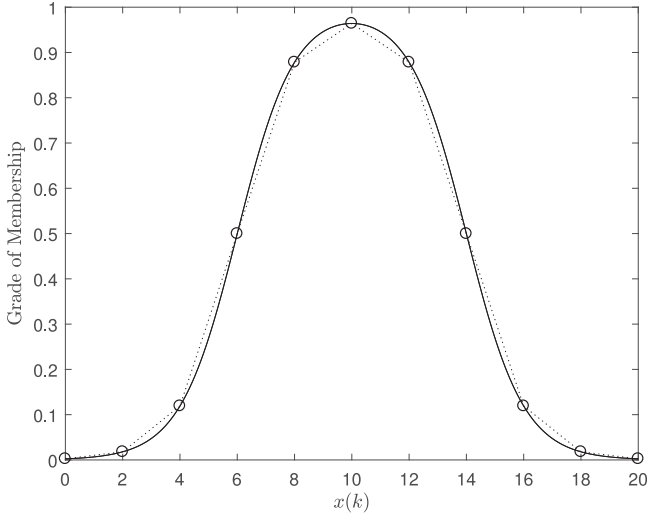


Fig. 1. Original membership functions (solid lines) and PLMFs (dotted lines).

represented as follows:

$$\hat{h}_{ij}(\mathbf{x}(k)) = \sum_{l=1}^D \delta_l(\mathbf{x}(k)) \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n p_{r i_r l}(x(k)) \hat{h}_{i_j i_1 i_2 \dots i_n l}(\mathbf{x}(k)) \quad (19)$$

where $\delta_l(\mathbf{x}(k))$ is a scalar index of subspaces, satisfying $\delta_l(\mathbf{x}(k)) = 1$, $\mathbf{x}(k) \in \varphi_l$, $l \in \underline{D} = \{1, 2, \dots, D\}$, otherwise $\delta_l(\mathbf{x}(k)) = 0$. $p_{r i_r l}(x(k))$ is the membership function corresponding to $\hat{h}_{i_j i_1 i_2 \dots i_n l}$, satisfying $0 \leq p_{r i_r l}(x(k)) \leq 1$ and $p_{r i_1 l}(x(k)) + p_{r i_2 l}(x(k)) = 1$. The membership function $p_{r i_r l}(x(k))$ is obtained as $p_{r 1 l}(x_r(k)) = \frac{x_{r 2 l}(k) - x_r(k)}{x_{r 2 l}(k) - x_{r 1 l}(k)}$ and $p_{r 2 l}(x_r(k)) = 1 - p_{r 1 l}(x_r(k))$.

To relax the stability conditions, the piecewise linear membership functions (19) are employed for stability analysis by considering the boundary information of approximation error $\delta_{ij} \leq w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) - \hat{h}_{ij}(\mathbf{x}(k)) \leq \bar{\delta}_{ij}$, where lower bounds $\underline{\delta}_{ij}$ and upper bounds $\bar{\delta}_{ij}$ are constants to be determined. Meanwhile, slack polynomial matrices $0 \leq \mathbf{Y}_{ij}(\mathbf{x}(k)) = [y_1^{ij}(\mathbf{x}(k)), y_2^{ij}(\mathbf{x}(k)), \dots, y_n^{ij}(\mathbf{x}(k))]^T \in \mathfrak{R}^n$ are used with the confining condition of $\mathbf{Y}_{ij}(\mathbf{x}(k)) \geq \mathbf{Q}_{ij}(\mathbf{x}(k))$. Therefore, (17) can be written as follows:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &\leq \sum_{i=1}^p \sum_{j=1}^c w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\mathbf{x}^T(k)\mathbf{Q}_{ij}(\mathbf{x}(k)) \\ &\leq \mathbf{x}^T(k) \left(\sum_{i=1}^p \sum_{j=1}^c \hat{h}_{ij}(\mathbf{x}(k))\mathbf{Q}_{ij}(\mathbf{x}(k)) \right. \\ &\quad \left. + \sum_{i=1}^p \sum_{j=1}^c (w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) \right. \\ &\quad \left. - \hat{h}_{ij}(\mathbf{x}(k)) - \underline{\delta}_{ij} + \bar{\delta}_{ij}) \times \mathbf{Q}_{ij}(\mathbf{x}(k)) \right) \end{aligned}$$

$$\begin{aligned} &= \mathbf{x}^T(k) \left(\sum_{i=1}^p \sum_{j=1}^c (\hat{h}_{ij}(\mathbf{x}(k)) + \underline{\delta}_{ij})\mathbf{Q}_{ij}(\mathbf{x}(k)) \right. \\ &\quad \left. + \sum_{i=1}^p \sum_{j=1}^c (w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) \right. \\ &\quad \left. - \hat{h}_{ij}(\mathbf{x}(k)) - \underline{\delta}_{ij})\mathbf{Q}_{ij}(\mathbf{x}(k)) \right) \\ &\leq \mathbf{x}^T(k) \left(\sum_{i=1}^p \sum_{j=1}^c (\hat{h}_{ij}(\mathbf{x}(k)) + \underline{\delta}_{ij})\mathbf{Q}_{ij}(\mathbf{x}(k)) \right. \\ &\quad \left. + \sum_{i=1}^p \sum_{j=1}^c (\bar{\delta}_{ij} - \underline{\delta}_{ij})\mathbf{Y}_{ij}(\mathbf{x}(k)) \right). \quad (20) \end{aligned}$$

It can be seen from (20) that $\Delta V(\mathbf{x}(k)) < 0$ is achieved if $\sum_{i=1}^p \sum_{j=1}^c (\hat{h}_{ij}(\mathbf{x}(k)) + \underline{\delta}_{ij})\mathbf{Q}_{ij}(\mathbf{x}(k)) + \sum_{i=1}^p \sum_{j=1}^c (\bar{\delta}_{ij} - \underline{\delta}_{ij})\mathbf{Y}_{ij}(\mathbf{x}(k)) < 0$ for all i and j . The stability analysis result is summarized in the following theorem.

Theorem 2: The discrete-time PFMB control system with time delay (12) is positive and asymptotically stable if there exist $\lambda \in \mathfrak{R}^n$ and $\mathbf{y}_k^j(\mathbf{x}(k)) \in \mathfrak{R}^m$ for $j \in \underline{c}$, $k \in \underline{n}$ such that Theorem 1 and the following SOS-based conditions are satisfied:

$$\lambda_r - \varepsilon_1 \text{ is SOS, } r \in \underline{n} \quad (21)$$

$$\begin{aligned} &- \left(\sum_{i=1}^p \sum_{j=1}^c (\hat{h}_{ij}(\mathbf{x}(k)) + \underline{\delta}_{ij})q_r^{ij}(\mathbf{x}(k)) \right. \\ &\quad \left. + \sum_{i=1}^p \sum_{j=1}^c (\bar{\delta}_{ij} - \underline{\delta}_{ij})y_r^{ij}(\mathbf{x}(k)) + \varepsilon_2(\mathbf{x}(k)) \right) \\ &\text{is SOS, } i \in \underline{p}, j \in \underline{c}, r \in \underline{n} \quad (22) \end{aligned}$$

$$y_r^{ij}(\mathbf{x}(k)) \text{ is SOS, } i \in \underline{p}, j \in \underline{c}, r \in \underline{n} \quad (23)$$

$$y_r^{ij}(\mathbf{x}(k)) - q_r^{ij}(\mathbf{x}(k)) \text{ is SOS, } i \in \underline{p}, j \in \underline{c}, r \in \underline{n} \quad (24)$$

where $\varepsilon_1 > 0$ is a predefined scalar and $\varepsilon_2(\mathbf{x}(k)) > 0$ is predefined scalar polynomial, $q_r^{ij}(\mathbf{x}(k))$ is defined in (18), and the feedback gains and the other variables are defined in Theorem 1. The decision variable λ is obtained by the SOS formulation satisfying Theorems 1 and 2. In order to impose the constraint $x_k \geq 0$, the technique of variable transformation is employed which simply turns x_k to x_k^2 , $k \in \underline{n}$.

Remark 5: The positivity and stability conditions based on Theorem 2 are only membership functions dependent positivity and stability conditions. This means the positivity and stability analyses merely consider the information of membership functions, not the information of premise variables.

D. Membership Functions and Premise Variables Dependent Stability Analysis

To further relax the SOS stability conditions based on non-PDC design concept (cf., Theorem 2), we must consider the upper and lower boundaries of premise variables. In this way,

the information of operating domain is incorporated in the stability conditions, resulting in more relaxed conditions. In this paper, we investigate the stability analysis based on a certain operating domain defined as $\mathbf{x}(k) \in [\mathbf{x}_1(k), \mathbf{x}_2(k)]$, where $\mathbf{x}_1(k)$ and $\mathbf{x}_2(k)$ are lower and upper bounds of the domain. We can then have the following constraint:

$$\sum_{e=1}^n (x_e(k) - x_{el}(k))(x_{e2}(k) - x_e(k))\mathbf{S}_e(\mathbf{x}(k)) \geq 0 \quad (25)$$

where $0 < \mathbf{S}_e(\mathbf{x}(k)) \in \mathfrak{R}^n$ is a polynomial expression.

From (25) and (20), we have

$$\begin{aligned} \Delta V(\mathbf{x}(k)) \leq & \mathbf{x}^T(k) \left(\sum_{i=1}^p \sum_{j=1}^c (\hat{h}_{ij}(\mathbf{x}(k)) + \underline{\delta}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}(k)) \right. \\ & + \sum_{i=1}^p \sum_{j=1}^c (\bar{\delta}_{ij} - \underline{\delta}_{ij}) \mathbf{Y}_{ij}(\mathbf{x}(k)) \\ & \left. + \sum_{r=1}^n (x_r(k) - x_{rl}(k))(x_{r2}(k) - x_r(k)) \mathbf{S}_e(\mathbf{x}(k)) \right). \end{aligned} \quad (26)$$

Therefore, the stability condition $\Delta V(\mathbf{x}(k)) < 0$ is achieved if $\sum_{i=1}^p \sum_{j=1}^c (\hat{h}_{ij}(\mathbf{x}(k)) + \underline{\delta}_{ij}) \mathbf{Q}_{ij}(\mathbf{x}(k)) + \sum_{i=1}^p \sum_{j=1}^c (\bar{\delta}_{ij} - \underline{\delta}_{ij}) \mathbf{Y}_{ij}(\mathbf{x}(k)) + \sum_{e=1}^n (x_e(k) - x_{el}(k))(x_{e2}(k) - x_e(k)) \mathbf{S}_e(\mathbf{x}(k)) < 0$ for all i and j , and we now propose the following theorem.

Theorem 3: The discrete-time PFMB control system with time delay (12) is positive and asymptotically stable if there exist $\lambda \in \mathfrak{R}^n$ and $\mathbf{y}_k^j(\mathbf{x}(k)) \in \mathfrak{R}^m$ for $j \in \underline{c}$, $k \in \underline{n}$ such that Theorem 1 and the following SOS-based conditions are satisfied:

$$\lambda_r - \varepsilon_1 \text{ is SOS, } r \in \underline{n} \quad (27)$$

$$\begin{aligned} - \left(\sum_{i=1}^p \sum_{j=1}^c (\hat{h}_{ij}(\mathbf{x}(k)) + \underline{\delta}_{ij}) q_r^{ij}(\mathbf{x}(k)) \right. \\ + \sum_{i=1}^p \sum_{j=1}^c (\bar{\delta}_{ij} - \underline{\delta}_{ij}) y_r^{ij}(\mathbf{x}(k)) \\ \left. + \sum_{e=1}^n (x_e(k) - x_{el}(k))(x_{e2}(k) - x_e(k)) s_r^e(\mathbf{x}(k)) + \varepsilon_2(\mathbf{x}(k)) \right) \\ \text{is SOS, } i \in \underline{p}, j \in \underline{c}, e, r \in \underline{n} \end{aligned} \quad (28)$$

$$y_r^{ij}(\mathbf{x}(k)) \text{ is SOS, } i \in \underline{p}, j \in \underline{c}, r \in \underline{n} \quad (29)$$

$$y_k^{ij}(\mathbf{x}(k)) - q_k^{ij}(\mathbf{x}(k)) \text{ is SOS, } i \in \underline{p}, j \in \underline{c}, k \in \underline{n} \quad (30)$$

$$s_r^e(\mathbf{x}(k)) \text{ is SOS, } e, r \in \underline{n} \quad (31)$$

where $\varepsilon_1 > 0$ is a predefined scalar and $\varepsilon_2(\mathbf{x}(k)) > 0$ is predefined scalar polynomial, $q_k^{ij}(\mathbf{x}(k))$ is defined in (18), and the feedback gains and the other variables are defined in Theorem 1. The decision variable λ is obtained by the SOS conditions in Theorems 1 and 3.

Remark 6: With introducing premise variables, the information of membership functions and premise variables are both considered in positivity and stability analyses in Theorem 3.

Remark 7: The positivity and stability conditions in Theorems 2 and 3 are independent from time delay. It means the positivity and stability are guaranteed without the values of time delays.

IV. SIMULATION EXAMPLE

In this section, a simulation example is provided to validate the proposed SOS-based stability and positivity conditions in Theorems 1–3 and check effectiveness of the proposed methods in relaxing the stability conditions. A discrete-time nonlinear plant with time delay and with system state matrix of $\mathbf{x}(k) = [x_1(k) \ x_2(k)]^T$ is considered with the following polynomial subsystems and input matrices under three fuzzy rules:

$$\begin{aligned} \mathbf{A}_{10}(x_1(k)) &= \begin{bmatrix} 0.02b + 0.4 + 0.015x_1(k) - 0.001x_1(k)^2 & 0.2 \\ & 0.2 \end{bmatrix} \\ \mathbf{A}_{20}(x_1(k)) &= \begin{bmatrix} 0.4 & 0.1 - 0.01x_1(k) \\ 0.2 & 0.1a \end{bmatrix} \\ \mathbf{A}_{30}(x_1(k)) &= \begin{bmatrix} 0.03 & 0.4 \\ 0.24 + 0.01x_1(k) & 0.06 + 0.0003x_1(k)^2 \end{bmatrix} \\ \mathbf{B}_1(x_1(k)) &= \begin{bmatrix} 0.2b + 0.1 \\ 0.1 - 0.001x_1(k)^2 \end{bmatrix} \\ \mathbf{B}_2(x_1(k)) &= \begin{bmatrix} 1 + 0.003x_1(k)^2 \\ 0.1 - 0.001x_1(k)^2 \end{bmatrix} \\ \mathbf{B}_3(x_1(k)) &= \begin{bmatrix} 1 + 0.005x_1(k)^2 \\ 0.1 - 0.001x_1(k)^2 \end{bmatrix} \\ \mathbf{A}_{11} = \mathbf{A}_{21} = \mathbf{A}_{31} &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix} \\ \mathbf{A}_{12} = \mathbf{A}_{22} = \mathbf{A}_{32} &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The parameters a and b are constant parameters chosen in the range of $0.37 \leq a \leq 8.21$ and $-0.14 \leq b \leq 3.5$ at the interval of 0.97 and 0.28, respectively.

The membership functions of polynomial fuzzy model are chosen as: $w_1(x_1(k)) = \mu_{M_1^1}(x_1(k)) = 1 - \frac{1}{1+e^{-(x_1(k)-6)}}$, $w_2(x_1(k)) = \mu_{M_1^2}(x_1(k)) = 1 - w_1(x_1(k)) - w_3(x_1(k))$, $w_3(x_1(k)) = \mu_{M_1^3}(x_1(k)) = \frac{1}{1+e^{-(x_1(k)-14)}}$.

Under the non-PDC design concept, the positivity and stability of the system is guaranteed based on a two-rule fuzzy controller. The membership functions of the fuzzy controller are chosen as follows:

$$m_1(x_1(k)) = \mu_{N_1^1}(x_1(k)) = \begin{cases} 1 & \text{for } x_1(k) < 3 \\ \frac{-x_1(k)+5}{10} & \text{for } 3 \leq x_1(k) \leq 17 \\ 0 & \text{for } x_1(k) > 17 \end{cases}$$

and

$$m_2(x_1(k)) = \mu_{N_1^2}(x_1(k)) = 1 - m_1(x_1(k)).$$

TABLE I
COMPARISON OF DIFFERENT INTERVALS OF SAMPLE POINTS AND DEGREES OF POLYNOMIAL CONTROLLER

Case	Interval	Degrees of $\mathbf{y}_k^j(\mathbf{x}(k))$	Sample points
1	5	4	$x_1(k) = \{0, 5, \dots, 15, 20\}$
2	5	5	$x_1(k) = \{0, 5, \dots, 15, 20\}$
3	2	4	$x_1(k) = \{0, 2, \dots, 18, 20\}$
4	2	5	$x_1(k) = \{0, 2, \dots, 18, 20\}$

To compare the positivity and stability regions for the discrete-time PFMB control system with time delay obtained from Theorems 2 to 3, piecewise linear membership functions (19) are employed to approximate the original membership functions and introduce the approximate error in the stability conditions. Table I compares the intervals of sample points of piecewise linear membership functions with the degrees of polynomial fuzzy controller, which influence the stability regions.

For the cases 1 and 2 with the interval of 5 for piecewise linear membership functions, the predefined constant scalars which represent the lower and upper boundaries of approximate error are obtained as follows:

$$\begin{aligned} \underline{\delta}_{11} &= -1.9452 \times 10^{-1}, & \underline{\delta}_{12} &= -6.2662 \times 10^{-2} \\ \underline{\delta}_{21} &= -9.2035 \times 10^{-2}, & \underline{\delta}_{22} &= -9.2035 \times 10^{-2} \\ \underline{\delta}_{31} &= -6.2662 \times 10^{-2}, & \underline{\delta}_{32} &= -1.9452 \times 10^{-1} \\ \bar{\delta}_{11} &= 1.7759 \times 10^{-1}, & \bar{\delta}_{12} &= 2.1816 \times 10^{-2} \\ \bar{\delta}_{21} &= 1.9796 \times 10^{-1}, & \bar{\delta}_{22} &= 1.9796 \times 10^{-1} \\ \bar{\delta}_{31} &= 2.1816 \times 10^{-2}, & \bar{\delta}_{32} &= 1.7759 \times 10^{-1}. \end{aligned}$$

In terms of cases 3 and 4 with the interval of 2 for piecewise linear membership functions, the predefined constant scalars which represent the lower and upper boundaries of approximate error are obtained as follow:

$$\begin{aligned} \underline{\delta}_{11} &= -4.2655 \times 10^{-2}, & \underline{\delta}_{12} &= -3.1457 \times 10^{-2} \\ \underline{\delta}_{21} &= -2.1857 \times 10^{-2}, & \underline{\delta}_{22} &= -2.1857 \times 10^{-2} \\ \underline{\delta}_{31} &= -3.1457 \times 10^{-2}, & \underline{\delta}_{32} &= -4.2655 \times 10^{-2} \\ \bar{\delta}_{11} &= 5.2626 \times 10^{-2}, & \bar{\delta}_{12} &= 1.9458 \times 10^{-2} \\ \bar{\delta}_{21} &= 4.2932 \times 10^{-2}, & \bar{\delta}_{22} &= 4.2932 \times 10^{-2} \\ \bar{\delta}_{31} &= 1.9458 \times 10^{-2}, & \bar{\delta}_{32} &= 5.2626 \times 10^{-2}. \end{aligned}$$

Initially, the positivity and stability conditions without the information of membership functions and premise variables are used to design the feedback gains of polynomial fuzzy controllers and check the system positivity and stability. This simply results in no positivity and stability regions. Then, we take into account the information of membership functions via the SOS stability conditions in Theorem 2. We set $\varepsilon_1 = \varepsilon_2(\mathbf{x}(k)) = 0.0010$, and $\mathbf{Y}(\mathbf{x}(k))$ as polynomial of degree 0 to 5 in $x_1(k)$. The resulting stability regions are shown in Fig. 2 where the stability region for case 1

TABLE II
POLYNOMIAL FUZZY CONTROLLER OBTAINED IN FIG. 2

Case	Theorem 2	
	a, b	Polynomial fuzzy controller
1	1.35, 0.42	$\mathbf{G}_1(x_1) = [0.4879 \times 10^{-5}x_1^4 - 0.0001x_1^3 + 0.0004x_1^2 + 0.0154x_1 + 0.0580, 0.8732 \times 10^{-6}x_1^4 + 0.1961 \times 10^{-4}x_1^3 - 0.0007x_1^2 + 0.0072x_1 + 0.0320]$
		$\mathbf{G}_2(x_1) = [0.4248 \times 10^{-5}x_1^4 - 0.0001x_1^3 + 0.0004x_1^2 + 0.0077x_1 + 0.0180, -0.2684 \times 10^{-6}x_1^4 - 0.4916 \times 10^{-5}x_1^3 - 0.0008x_1^2 + 0.0034x_1 + 0.1587]$
2	0.37, 0.14	$\mathbf{G}_1(x_1) = [0.1022 \times 10^{-9}x_1^5 + 0.7770 \times 10^{-5}x_1^4 - 0.0001x_1^3 + 0.5028 \times 10^{-4}x_1^2 + 0.0189x_1 + 0.0844, -0.2501 \times 10^{-9}x_1^5 + 0.1377 \times 10^{-5}x_1^4 + 0.4017 \times 10^{-4}x_1^3 - 0.0007x_1^2 + 0.0075x_1 + 0.0319]$
		$\mathbf{G}_2(x_1) = [-0.4126 \times 10^{-10}x_1^5 + 0.6398 \times 10^{-5}x_1^4 - 0.0001 \times 10^{-3}x_1^3 + 0.0002x_1^2 + 0.0119x_1 + 0.0303, 0.4323 \times 10^{-10}x_1^5 + 0.1500 \times 10^{-5}x_1^4 + 0.5329 \times 10^{-4}x_1^3 - 0.0012x_1^2 + 0.0028x_1 + 0.0475]$
Case	Theorem 3	
	a, b	Polynomial fuzzy controller
1	3.31, 0.42	$\mathbf{G}_1(x_1) = [0.1542 \times 10^{-5}x_1^4 - 0.4718 \times 10^{-4}x_1^3 + 0.0023x_1^2 - 0.0239x_1 + 0.0839, 0.2371 \times 10^{-6}x_1^4 - 0.1062 \times 10^{-4}x_1^3 + 0.0003x_1^2 - 0.0020x_1 + 0.0423]$
		$\mathbf{G}_2(x_1) = [0.1499 \times 10^{-5}x_1^4 - 0.4325 \times 10^{-4}x_1^3 + 0.0020x_1^2 - 0.0268x_1 + 0.1202, 0.2564 \times 10^{-6}x_1^4 - 0.4784 \times 10^{-5}x_1^3 - 0.0008x_1^2 + 0.0031x_1 + 0.1600]$
2	3.31, 0.14	$\mathbf{G}_1(x_1) = [0.1359 \times 10^{-12}x_1^5 + 0.3540 \times 10^{-5}x_1^4 - 0.0001x_1^3 + 0.0025 \times 10^{-4}x_1^2 - 0.0181x_1 + 0.0769, -0.1425 \times 10^{-12}x_1^5 + 0.6530 \times 10^{-6}x_1^4 + 0.3390 \times 10^{-4}x_1^3 + 0.0009x_1^2 - 0.0029x_1 + 0.0419]$
		$\mathbf{G}_2(x_1) = [0.1431 \times 10^{-14}x_1^5 + 0.3615 \times 10^{-5}x_1^4 - 0.0001x_1^3 + 0.0018x_1^2 - 0.0138x_1 + 0.0940, 0.1017 \times 10^{-12}x_1^5 + 0.3170 \times 10^{-6}x_1^4 + 0.4415 \times 10^{-5}x_1^3 - 0.0007x_1^2 + 0.0017x_1 + 0.1390]$

TABLE III
 POLYNOMIAL FUZZY CONTROLLER OBTAINED IN FIG. 3

Case	Theorem 2	
	a, b	Polynomial fuzzy controller
3	8.21, 0.14	$\mathbf{G}_1(x_1) = [0.7717 \times 10^{-5}x_1^4 - 0.0001x_1^3$ $+ 0.0002x_1^2 + 0.0022x_1 - 0.0060,$ $0.6986 \times 10^{-6}x_1^4 + 0.3046 \times 10^{-4}x_1^3$ $- 0.0001x_1^2 + 0.0050x_1 - 0.0597]$
		$\mathbf{G}_2(x_1) = [0.6193 \times 10^{-5}x_1^4 - 0.0001 \times$ $10^{-4}x_1^3 + 0.0002x_1^2 + 0.0050x_1 - 0.0125,$ $0.9563 \times 10^{-6}x_1^4 + 0.4941 \times 10^{-4}x_1^3$ $- 0.0005x_1^2 + 0.0019x_1 - 0.0265]$
4	8.21, 0.14	$\mathbf{G}_1(x_1) = [0.3087 \times 10^{-13}x_1^5 + 0.9795 \times$ $10^{-5}x_1^4 - 0.0001x_1^3 - 0.0002x_1^2$ $+ 0.0021x_1 + 0.0053, 0.3728 \times 10^{-13}x_1^5$ $+ 0.7289 \times 10^{-6}x_1^4 + 0.2182 \times 10^{-4}x_1^3$ $- 0.0001x_1^2 + 0.0064x_1 - 0.0692]$
		$\mathbf{G}_2(x_1) = [0.4493 \times 10^{-12}x_1^5 + 0.8219 \times$ $10^{-5}x_1^4 - 0.0001x_1^3 - 0.0004x_1^2 +$ $0.0058x_1 + 0.0062, -0.8196 \times 10^{-13}x_1^5$ $+ 0.4387 \times 10^{-6}x_1^4 + 0.2894 \times 10^{-4}x_1^3$ $- 0.0004x_1^2 + 0.0055x_1 - 0.0532]$
Case	Theorem 3	
	a, b	Polynomial fuzzy controller
3	8.21, -0.14	$\mathbf{G}_1(x_1) = [0.8927 \times 10^{-5}x_1^4 - 0.0002x_1^3$ $+ 0.0027x_1^2 - 0.0108x_1 - 0.0035,$ $0.4196 \times 10^{-6}x_1^4 - 0.2605 \times 10^{-4}x_1^3$ $+ 0.0005x_1^2 + 0.0071x_1 - 0.0880]$
		$\mathbf{G}_2(x_1) = [0.8763 \times 10^{-5}x_1^4 - 0.0002 \times$ $10^{-4}x_1^3 + 0.0017x_1^2 - 0.0034x_1 - 0.0078,$ $0.2069 \times 10^{-6}x_1^4 + 0.3297 \times 10^{-5}x_1^3$ $- 0.8361 \times 10^{-4}x_1^2 + 0.0072x_1 - 0.0709]$
4	8.21, -0.14	$\mathbf{G}_1(x_1) = [-0.1314 \times 10^{-11}x_1^5 + 0.8927 \times$ $10^{-5}x_1^4 - 0.0002x_1^3 + 0.0027x_1^2$ $- 0.0108x_1 - 0.0035, -0.7436 \times 10^{-12}x_1^5$ $+ 0.4196 \times 10^{-6}x_1^4 + 0.2605 \times 10^{-4}x_1^3$ $+ 0.0005x_1^2 + 0.0071x_1 - 0.0880]$
		$\mathbf{G}_2(x_1) = [-0.2630 \times 10^{-11}x_1^5 + 0.8763 \times$ $10^{-5}x_1^4 - 0.0002x_1^3 + 0.0017x_1^2 -$ $0.0034x_1 - 0.0078, -0.1770 \times 10^{-11}x_1^5$ $- 0.2069 \times 10^{-6}x_1^4 + 0.3297 \times 10^{-5}x_1^3$ $- 0.8361 \times 10^{-4}x_1^2 + 0.0072x_1 - 0.0709]$

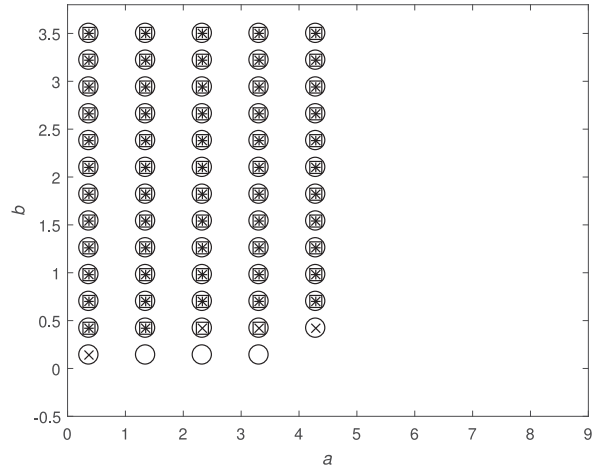


Fig. 2. Stability regions given by Theorem 2 indicated by “+” for case 1, “x” for case 2, given by Theorem 3 indicated by “□” for case 1, “o” for case 2.

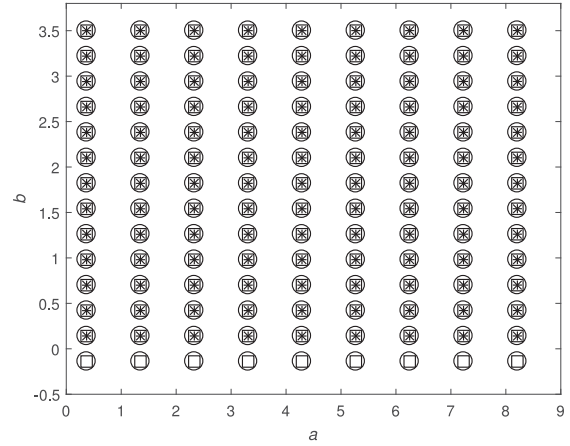


Fig. 3. Stability regions given by Theorem 2 indicated by “+” for case 3, “x” for case 4, given by Theorem 3 indicated by “□” for case 3, “o” for case 4.

is shown with “+,” and for case 2 is shown with “x.” In the next step, using Theorem 3, we consider both membership functions and premise variables for the controller design and stability analysis. We set the boundary of premise variable as $\mathbf{x}(k) \in [0, 20]$, $\varepsilon_1 = \varepsilon_2(\mathbf{x}(k)) = 0.0010$, $\mathbf{Y}(\mathbf{x}(k))$ as a polynomial of degree 0 to 5 in x_1 , and $\mathbf{S}_e(\mathbf{x}(k))$ as a polynomial of degree 0 to 3 in $x_1(k)$. The resulting stability regions are shown in Fig. 2, where the stability region for case 1 is shown by “□” and for case 2 is shown by “o.”

Using the same aforementioned setting, we apply Theorem 2 for cases 3 and 4 (the stability regions are shown in Fig. 3 with “+” sign for case 3, and “x” sign for case 4), and then Theorem 3 for the same cases (the stability regions are shown in Fig. 3 with “□” sign for case 3 and “o” sign for case 4).

As noticed in the positivity and stability regions in Figs. 2 and 3, we can achieve smaller interval of sample points of membership functions and higher degrees of polynomial fuzzy controllers leading to larger stabilization region. The smaller interval of the sampled points make the lower limit $\bar{\delta}_{ij}$ and the upper limit $\bar{\delta}_{ij}$ of the approximation error smaller. This proves

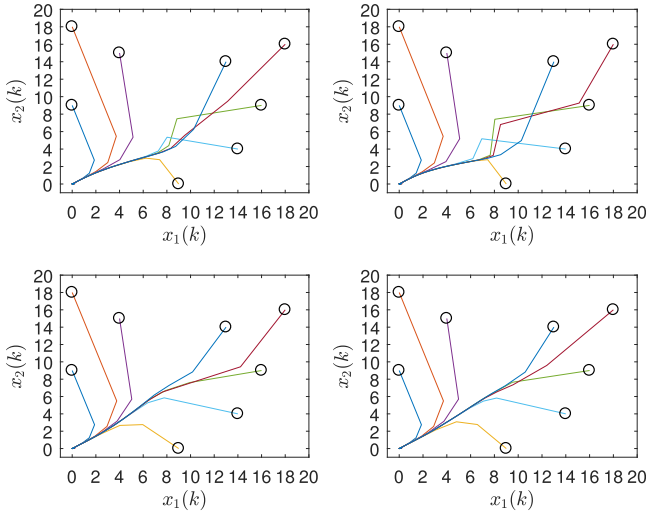


Fig. 4. Top left phase plots are $a = 1.35; b = 0.42$ (stability regions indicated by the symbols “+” for case 1), the top right phase plots are $a = 0.37; b = 0.14$ (stability region indicated by the symbols “x” for case 2). The bottom left phase plots are $a = 3.31; b = 0.42$ (stability regions indicated by the symbols “□” for case 1), the bottom right phase plots are $a = 3.31; b = 0.14$ (stability region indicated by the symbols “o” for case 2) all with referring to Fig. 2 and time delay $\tau_1 = 50, \tau_2 = 100$.

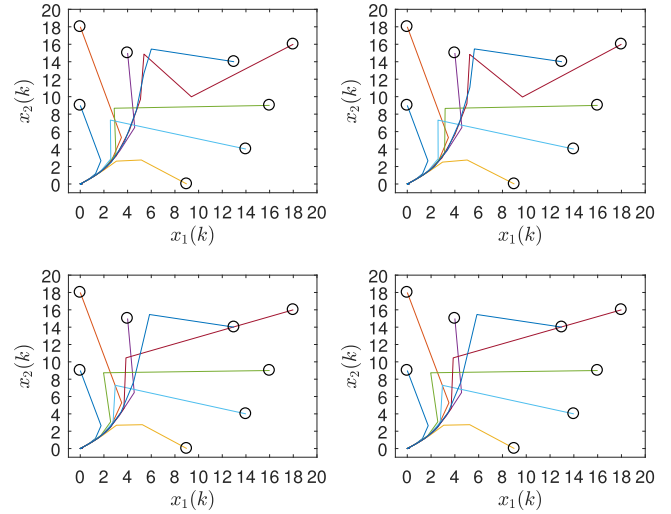


Fig. 6. Top left phase plots are $a = 8.21; b = 0.14$ (stability regions indicated by the symbols “+” for case 3), the top right phase plots are $a = 8.21; b = 0.14$ (stability region indicated by the symbols “x” for case 4). The bottom left phase plots are $a = 8.21; b = -0.14$ (stability regions indicated by the symbols “□” for case 3), the bottom right phase plots are $a = 8.21; b = -0.14$ (stability region indicated by the symbols “o” for case 4) all with referring to Fig. 3 and time delay $\tau_1 = 50, \tau_2 = 100$.

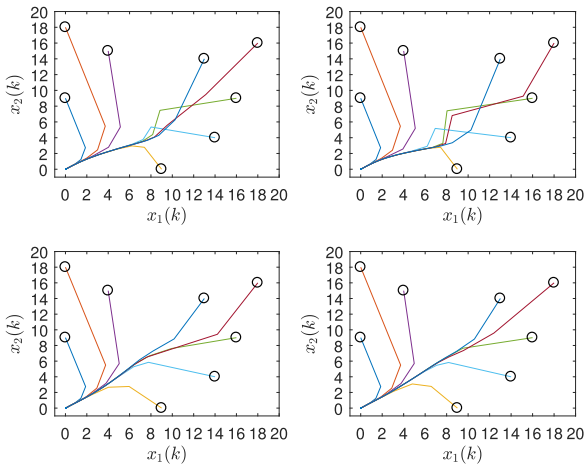


Fig. 5. Top left phase plots are $a = 1.35; b = 0.42$ (stability regions indicated by the symbols “+” for case 1), the top right phase plots are $a = 0.37; b = 0.14$ (stability region indicated by the symbols “x” for case 2). The bottom left phase plots are $a = 3.31; b = 0.42$ (stability regions indicated by the symbols “□” for case 1), the bottom right phase plots are $a = 3.31; b = 0.14$ (stability region indicated by the symbols “o” for case 2) all with referring to Fig. 2 and time delay $\tau_1 = 100, \tau_2 = 200$.

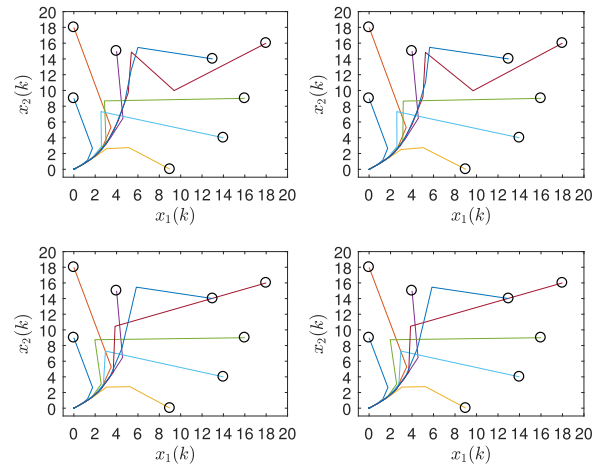


Fig. 7. Top left phase plots are $a = 8.21; b = 0.14$ (stability regions indicated by the symbols “+” for case 3), the top right phase plots are $a = 8.21; b = 0.14$ (stability region indicated by the symbols “x” for case 4). The bottom left phase plots are $a = 8.21; b = -0.14$ (stability regions indicated by the symbols “□” for case 3), the bottom right phase plots are $a = 8.21; b = -0.14$ (stability region indicated by the symbols “o” for case 4) all with referring to Fig. 3 and time delay $\tau_1 = 100, \tau_2 = 200$.

the fact that more information of the membership functions is introduced in the stability conditions, and the flexibility of polynomial fuzzy controllers is enhanced by adopting higher degrees of polynomial fuzzy controllers, which all results in larger stability region.

It is visible in Figs. 2 and 3 that the stability regions obtained by Theorem 3 are larger than what are obtained in Theorem 2 which only takes into account the information of membership functions. Piecewise linear membership functions are only able to carry limited information of membership functions in Theorem 2. Boundary information of premise variables

contain rich nonlinearity information of the positivity and stability. Therefore, we must consider premise variables in addition to the membership functions to obtain more relaxed positivity stability conditions.

To verify the obtained positivity and stability regions, the phase plots corresponding to each situation is shown in Figs. 2 and 3. The stability results are shown in Figs. 4–7. The corresponding polynomial fuzzy controllers are listed in Tables II and III. To ensure the validity of results, the phase plots of $x_1(k)$ and $x_2(k)$ are simulated with eight different initial conditions indicated by “o.” These results as noticed from Figs. 4 to 7

prove that the polynomial fuzzy controller is able to drive all the system states to equilibrium (origin) while always hold them positive based on different initial conditions. When we change the values of time delays $\tau_1 = 50$ to $\tau_1 = 100$ (cf., Figs. 4 and 6), and $\tau_2 = 100$ to $\tau_2 = 200$ (cf., Figs. 5 and 7), the system become stable regardless of the values of time delays. This is mainly because the stability conditions in Theorems 2–3 are independent of delay period.

V. CONCLUSION

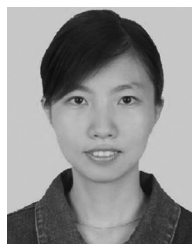
This paper proposed a novel Lyapunov approach to reduce the conservativeness of positivity and stability conditions, which are used for the stability analysis and controller design of discrete-time PFMB control systems with time delay. To automate the whole analysis as a feasibility problem, all the Lyapunov stability conditions and the non-PDC design of controller gains are formulated as SOS conditions. With the new approach, valuable nonlinearity information which exists in membership functions and premise variables can be incorporated into the stability analysis. In this respect, a piecewise linear membership function framework is proposed to approximate the original membership functions and introduce the resulting approximation error in the stability analysis where the boundary information of premise variables are additionally introduced in the form of slack matrices. The prowess of the proposed theorems in relaxing the stability conditions have been presented via a case study where all the stability regions are extended and the controller can make the system stable regardless of the system time delay.

In the future, by considering the property of membership functions, e.g., $\sum_{i=1}^p w_i(\mathbf{x}(k)) = 1$, $0 < w_i(\mathbf{x}(k)) < 1$, and the boundary of membership functions in the proposed stability Theorems, we can further relax the stability conditions. Moreover, to further reduce the conservativeness of the proposed stability conditions, the Lyapunov function candidate with the polynomial form of λ should be explored. The proposed membership and premise variable-dependent stability analysis can be used as the base for developing output-feedback and observer-based feedback controller, where the direct measurement of system states is practically difficult.

REFERENCES

- [1] Y. Ebihara, D. Peaucelle, and D. Arzelier, "LMI approach to linear positive system analysis and synthesis," *Syst. Control Lett.*, vol. 63, no. 1, pp. 50–56, 2014.
- [2] Y. Ebihara, D. Peaucelle, and D. Arzelier, "Analysis and synthesis of interconnected positive systems," *IEEE Trans. Autom. Control*, vol. 62, no. 2, pp. 652–667, Feb. 2017.
- [3] F. Forni and R. Sepulchre, "Differentially positive systems," *IEEE Trans. Autom. Control*, vol. 61, no. 2, pp. 346–359, Feb. 2016.
- [4] J. Liu, K. Zhang, G. Pang, and H. Wei, "Robust stabilisation for constrained discrete-time switched positive linear systems with uncertainties," *IET Control Theory Appl.*, vol. 9, no. 17, pp. 2598–2605, 2015.
- [5] L. Benvenuti and L. Farina, "Positive and compartmental systems," *IEEE Trans. Autom. Control*, vol. 47, no. 2, pp. 370–373, Feb. 2002.
- [6] Y. Ebihara, D. Peaucelle, and D. Arzelier, "Steady-state analysis of delay interconnected positive systems and its application to formation control," *Dong*, vol. 11, no. 16, pp. 2783–2792, 2017.
- [7] J. Wang and J. Zhao, "Stabilisation of switched positive systems with actuator saturation," *IET Control Theory Appl.*, vol. 10, no. 6, pp. 717–723, 2016.
- [8] J. G. Dong, "On the decay rates of homogeneous positive systems of any degree with time-varying delays," *IEEE Trans. Autom. Control*, vol. 60, no. 11, pp. 2983–2988, Nov. 2015.
- [9] H. Yang and Y. Zhang, "Stability of positive delay systems with delayed impulses," *IET Control Theory Appl.*, vol. 12, no. 2, pp. 194–205, 2018.
- [10] M. A. Rami and D. Napp, "Discrete-time positive periodic systems with state and control constraints," *IEEE Trans. Autom. Control*, vol. 61, no. 1, pp. 234–239, Jan. 2016.
- [11] P. T. Nam, H. Trinh, P. N. Pathirana, and V. N. Phat, "Stability analysis of nonlinear time-delay systems using a novel piecewise positive systems method," *IEEE Trans. Autom. Control*, vol. 63, no. 1, pp. 291–297, Jan. 2018.
- [12] T. Wang, J. Qiu, and H. Gao, "Adaptive neural control of stochastic nonlinear time-delay systems with multiple constraints," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 8, pp. 1875–1883, Aug. 2017.
- [13] M. M. Haddad and V. Chellaboina, "Stability theory for nonnegative and compartmental dynamical systems with time delay," in *Proc. Amer. Control Conf.*, Jun. 2004, vol. 2, pp. 1422–1427.
- [14] Y. Ariba and F. Gouaisbaut, "Construction of Lyapunov–Krasovskii functional for time-varying delay systems," in *Proc. IEEE Conf. Decis. Control*, Dec. 2008, pp. 3995–4000.
- [15] H.-N. Wu, "Delay-dependent stability analysis and stabilization for discrete-time fuzzy systems with state delay: A fuzzy Lyapunov–Krasovskii functional approach," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 36, no. 4, pp. 954–962, Aug. 2006.
- [16] Y. Mao, H. Zhang, Y. Qin, and C. Dang, "Stability and constrained control of a class of discrete-time fuzzy positive systems with time-varying delays," *Circuits, Syst., Signal Process.*, vol. 32, no. 2, pp. 889–904, 2013.
- [17] Y. Mao, H. Zhang, and C. Dang, "Stability analysis and constrained control of a class of fuzzy positive systems with delays using linear copositive Lyapunov functional," *Circuits, Syst., Signal Process.*, vol. 31, no. 5, pp. 1863–1875, 2012.
- [18] X. Liu, W. Yu, and L. Wang, "Stability analysis for continuous-time positive systems with time-varying delays," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 1024–1028, Apr. 2010.
- [19] A. Benzaouia, R. Oubah, and A. El Hajjaji, "Stabilization of positive Takagi–Sugeno fuzzy discrete-time systems with multiple delays and bounded controls," *J. Franklin Inst.*, vol. 351, no. 7, pp. 3719–3733, 2014.
- [20] X. Chen, J. Lam, and H. Lam, "Positive filtering for positive Takagi–Sugeno fuzzy systems under $l(1)$ performance," *Inf. Sci.*, vol. 299, pp. 32–41, 2015.
- [21] Y. Wu, H. Luo, and H. Zhang, "Stability analysis of discrete-time fuzzy positive systems with time delays," *J. Intell. Fuzzy Syst.*, vol. 25, no. 4, pp. 893–905, 2013.
- [22] A. Benzaouia, A. Hmamed, and A. El Hajjaji, "Stabilization of controlled positive discrete-time T–S fuzzy systems by state feedback control," *Int. J. Adaptive Control Signal Process.*, vol. 24, no. 12, pp. 1091–1106, 2010.
- [23] A. Benzaouia, A. E. Hajjaji, and A. Hmamed, "Relaxed stabilization of controlled positive discrete-time T–S fuzzy systems," in *Proc. 18th Mediterranean Conf. Control Automat.*, Jun. 2010, pp. 1407–1412.
- [24] X. Liu, "Stability and stabilization of positive systems with delays," in *Proc. Int. Conf. Commun., Circuits Syst.*, Jul. 2009, pp. 908–912.
- [25] M. Fadali and S. Jafarzadeh, "Stability analysis of positive interval type-2 TSK systems with application to energy markets," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 1031–1038, Aug. 2014.
- [26] H. Feyzmahdavian, T. Charalambous, and M. Johansson, "Exponential stability of homogeneous positive systems of degree one with time-varying delays," *IEEE Trans. Autom. Control*, vol. 59, no. 6, pp. 1594–1599, Jun. 2014.
- [27] P. Li, J. Lam, and Z. Shu, " H_∞ positive filtering for positive linear discrete-time systems: An augmentation approach," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2337–2342, Oct. 2010.
- [28] X. Li, C. Liu, H. K. Lam, F. Liu, and X. Zhao, "Stability analysis of fuzzy polynomial positive systems with time delay," in *Proc. Int. Conf. Fuzzy Theory Appl.*, Nov. 2014, pp. 24–28.
- [29] X. Li, H. K. Lam, F. Liu, and X. Zhao, "Stability and stabilization analysis of positive polynomial fuzzy systems with time delay considering piecewise membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 4, pp. 958–971, Aug. 2017.
- [30] X. Li, H. K. Lam, G. Song, and F. Liu, "Stability analysis of positive polynomial fuzzy-model-based control systems with time delay under imperfect premise matching," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 4, pp. 2289–2300, Aug. 2018.
- [31] H. K. Lam and S. H. Tsai, "Stability analysis of polynomial-fuzzy-model-based control systems with mismatched premise membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 1, pp. 223–229, Feb. 2014.

- [32] H. K. Lam, C. Liu, L. Wu, and X. Zhao, "Polynomial fuzzy-model-based control systems: Stability analysis via approximated membership functions considering sector nonlinearity of control input," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 6, pp. 2202–2214, Dec. 2015.
- [33] H. K. Lam, "Polynomial fuzzy-model-based control systems: Stability analysis via piecewise-linear membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 3, pp. 588–593, Jun. 2011.
- [34] G. Song, H. K. Lam, and X. Yang, "Membership-function-dependent stability analysis of interval type-2 polynomial fuzzy-model-based control systems," *IET Control Theory Appl.*, vol. 11, no. 17, pp. 3156–3170, 2017.
- [35] Y. Li, H. K. Lam, and L. Zhang, "Control design for interval type-2 polynomial fuzzy-model-based systems with time-varying delay," *IET Control Theory Appl.*, vol. 11, no. 14, pp. 2270–2278, 2017.
- [36] M. Rakhshan, N. Vafamand, M. H. Khooban, and F. Blaabjerg, "Maximum power point tracking control of photovoltaic systems: A polynomial fuzzy model-based approach," *IEEE J. Emerg. Sel. Topics Power Electron.*, vol. 6, no. 1, pp. 292–299, Mar. 2018.
- [37] A. Chibani, M. Chadli, and N. B. Braiek, "A sum of squares approach for polynomial fuzzy observer design for polynomial fuzzy systems with unknown inputs," *Int. J. Control Autom. Syst.*, vol. 14, no. 1, pp. 323–330, 2016.
- [38] A. Chibani, M. Chadli, S. X. Ding, and N. B. Braiek, "Design of robust fuzzy fault detection filter for polynomial fuzzy systems with new finite frequency specifications," *Automatica*, vol. 93, pp. 42–54, 2018.
- [39] Y. Chen, H. Ohtake, K. Tanaka, and W. Wang, "Stability analysis for the polynomial fuzzy systems by utilizing equality constraints of sum-of-squares program," in *Proc. Int. Conf. Syst. Sci. Eng.*, Jul. 2010, pp. 36–40.
- [40] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOS-TOOLS: A general purpose sum of squares programming solver," in *Proc. 41st IEEE Conf. Decis. Control*, Dec. 2002, vol. 1, pp. 741–746.
- [41] K. Tanaka, T. Ikeda, and H. O. Wang, "Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs," *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 2, pp. 250–265, May 1998.
- [42] Y. Zhao, H. K. Lam, G. Song, and X. Yin, "Relaxed stability conditions for polynomial-fuzzy-model-based control system with membership function information," *IET Control Theory Appl.*, vol. 11, no. 10, pp. 1493–1502, 2017.
- [43] H. Hassani, J. Zarei, M. Chadli, and J. Qiu, "Unknown input observer design for interval type-2 T-S fuzzy systems with immeasurable premise variables," *IEEE Trans. Cybern.*, vol. 47, no. 9, pp. 2639–2650, Sep. 2017.
- [44] H. K. Lam and L. D. Seneviratne, "Stability analysis of polynomial fuzzy-model-based control systems under perfect/imperfect premise matching," *IET Control Theory Appl.*, vol. 5, no. 15, pp. 1689–1697, Oct. 2011.
- [45] H. K. Lam, "Design of stable fuzzy controller for non-linear systems subject to imperfect premise matching based on grid-point approach," *IET Control Theory Appl.*, vol. 4, no. 12, pp. 2770–2780, Dec. 2010.



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