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Membership-dependent stability analysis of discrete-time positive polynomial fuzzy-model-based control systems with time delay

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Abstract: This study proposes a novel, relaxed, Lyapunov-based and membership-function dependent stabilisation analysis of discrete-time polynomial-fuzzy-model-based (PFMB) control systems with time delay under positivity constraint. The discrete-time non-linear system with time delay is represented by a polynomial fuzzy model, and corresponding PFMB controller is designed using imperfect premise matching technique which does not require its fuzzy rule and shape of membership functions be matched with those of the model. The authors take advantage of this property to relax the conservativeness of the obtained stability results by introducing the information of membership functions, i.e. the relationship constraint information of membership functions between the model and the controller and boundary information of membership functions, into the stability and positivity conditions. A numerical example is given to demonstrate the effectiveness of the proposed approach.

1 Introduction

Positive systems are referred to as a kind of systems whose state variables always confine to be positive whenever the initial condition is non-negative. We find examples of positive systems in a number of industrial processes including the one in chemical reactor, heat exchanges and storage systems [1–4]. The stability analysis research related to different types positive systems are fruitful, such as interval positive systems in [5], switched positive systems in [6–8] and Markov Jump positive systems in [9]. In addition, the dynamics of a non-linear system in the real world commonly involves time delay which may lead the system to unsatisfactory performance and even instability [10–12]. There are many research achievements of time delay systems, such as filter design in [13], predictive control in [14] and output feedback control in [15]. Therefore, it is meaningful to take into account the presence of time delay in the stability analysis and controller design for positive non-linear systems.

Many industrial processes are described by discrete time non-linear models, and fuzzy control is an effective way to deal with the discrete time non-linear systems. Different methods of fuzzy control of discrete time non-linear systems have been conducted such as stabilisation issue in [16] and observer design in [17]. For the modelling of such systems, instead of using the popular Takagi–Sugeno (T–S) fuzzy model [7, 8, 18, 19], we employ polynomial fuzzy model whose subsystems are weighted by polynomial terms with corresponding membership functions to represent the dynamics of discrete time non-linear positive system with time delay. The reason is that compared with T–S fuzzy model, polynomial fuzzy model can represent wider non-linear systems [20–22]. In the polynomial fuzzy model, the original non-linear components of the polynomial terms are kept intact which in turn decreases the number of fuzzy rules. Most importantly, due to the capability of the precise approximation, stability analysis and control synthesis based on the polynomial fuzzy model are more applicable to the original non-linear system when compared with T–S fuzzy model. In terms of computation complexity, compared with T–S fuzzy model, although the formulation of system, input and time delay matrices of polynomial fuzzy model is comparatively complicated, the simplified structure of polynomial fuzzy model and less number of fuzzy rules will also reduce computation complexity. Based on polynomial fuzzy model, a polynomial fuzzy controller then can be designed to guarantee the stability and positivity of discrete-time polynomial-fuzzy-model-based (PFMB) control system with time delay. The stability analysis along with the controller design is formulated as sum-of-squares (SOS) feasibility and positivity solutions which can be solved by the third party MATLAB toolbox SOSTOOLS [20–23]. The advantage of SOSTOOLS is the existence of polynomial terms in the stability conditions which cannot be solved by LMI toolbox but can be solved by SOSTOOLS, and polynomial fuzzy controller is allowed to be realised through SOSTOOLS.

The novel imperfect premise matching (IPC) design concept proposed by Lam [24, 25] introduced a breakthrough in the traditional PDC design concept [26–28] in designing fuzzy controller. Compared with traditional PDC design, IPC design concept can improve the flexibility of fuzzy controller where the shape of membership functions and number of fuzzy rules of the controller can be independent of those of the model [29–32]. Based on IPC design concept, a novel idea of membership-function dependent analysis [24, 25] is proposed to bring the membership function information into stability/performance/robustness analysis. This leads to further relaxation of the conservativeness of the SOS formulation. The information of membership functions can be introduced using local/regional membership function information [24, 25, 32], staircase membership functions [33], piecewise linear membership functions [34] and Taylor series membership functions [21].

For the stability analysis of discrete-time positive PFMB control system with time delay, efforts can be made following mainly two directions. For the first direction, favourable Lyapunov function candidate is employed to reduce the conservativeness of stability analysis of discrete time positive PFMB control system with time delay. In this paper, instead of quadratic Lyapunov–Krosovskill function frequently used in the literature [28, 35, 36], a new form of Lyapunov function candidate, i.e. linear co-positive Lyapunov function (LCLF) is employed. LCLF not only follows the basic principle of Lyapunov stability theory but considers the nature of positivity in discrete-time positive PFMB control system with time delay [27, 37]. For the second direction, IPC design concept and membership function dependent stability analysis [24, 25] are applied to relax the stability analysis of discrete time positive PFMB control system with time delay. Few attempts have shown success in relaxing stability of the positive PFMB control system with time delay [37–40]. In [37–39] by considering membership functions as symbolic variables or piecewise membership functions, the stability and positivity can be achieved. In this way, the information of membership functions can be introduced into the stability conditions to reduce their conservativeness formulation.

In this paper, we propose a novel method to include the information of membership functions by the relationship constraint information between membership functions of the fuzzy model and controller in the overall state space and introduce it into the stability analysis by slack matrices. In a number of membership function dependent stability analysis literature [20, 37–39], by considering the lower and upper boundary of membership functions of fuzzy model $w_i(\mathbf{x}(k))$ and fuzzy controller $m_j(\mathbf{x}(k))$ denoting $\underline{\eta}_i \le w_i(\mathbf{x}(k)) \le \overline{\eta}_i$, $\underline{\varphi}_j \le m_j(\mathbf{x}(k)) \le \overline{\varphi}_j$, the information regarding shape of the membership functions is introduced into the stability analysis. In this paper, this procedure will be generalised in a way that any constraint relationship information on the membership function shape as $\sigma_{ij} w_i(\mathbf{x}(k)) m_j(\mathbf{x}(k)) + \ell_i w_i(\mathbf{x}(t)) + \Gamma_j m_j(\mathbf{x}(k)) - \varphi \ge 0$ can be incorporated in the SOS-based stability conditions. As a result, the conservativeness of stability and positivity conditions of discrete time PFMB control system with time delay can be reduced based on constraint relationship information of the membership functions. Furthermore, unlike boundary of membership functions which carries limited information of membership functions, arbitrary number of constraints of membership functions between model and controller can be given and integrated into the stability analysis. This means that one can arbitrarily increase the amount of information of membership functions into the stability and positivity conditions. Together with constraint information of membership functions between model and controller, the boundary information of the membership functions of model and controller will also be included in the stability analysis. Furthermore, as the information of the membership functions is carried by some slack matrices to the stability analysis and the number of stability conditions are generally higher, computational demand on finding a feasible solution to the stability conditions will be higher, however, in the membership function dependent stability analysis, the stability conditions obtained are not for any shape of membership functions but dedicated to the fuzzy model based control system to be controlled, which means we can much more easily find the wanted fuzzy controller to stabilise the non-linear system. Therefore, the conservativeness of stability analysis can be reduced further. The contributions of this paper are listed as below:

1. We use the relationship constraint information between membership functions of polynomial fuzzy model and controller along with the boundary information of the membership functions to relax the (membership-function dependent) stability and positivity analysis of a discrete time PFMB control system with time delay, which means arbitrary number of constraints of membership functions between model and controller can be given and integrated into the stability analysis.
2. Polynomial fuzzy model is proposed to represent the dynamics of a discrete-time PFMB control system with time delay,
3. IPC design concept is used to design the polynomial fuzzy controller and formulate the stability/positivity feasibility conditions, which means the number of rules and membership functions of the polynomial fuzzy controller can be freely chosen (i.e. $c \ne p, m_1, \ldots, m_c \ne w_1, \ldots, w_p$) independently of the polynomial fuzzy model.

The rest of paper is organised as follows. In Section 2, the formulation of discrete-time polynomial fuzzy model and controller are provided. In Section 3, relaxed stability/positivity conditions are proposed in form of two theorems. The conditions include the relationship constraint information between membership functions of the polynomial fuzzy model and controller along with the boundary information of the membership functions. In Section 4, a simulation example is given to validate the proposed theoretical analysis. Paper is concluded in Section 5.

2 Notations and preliminaries

2.1 Notation

The following notations are adopted throughout this paper. A monomial in $\mathbf{x}(k) = [x_1(k), x_2(k), \ldots, x_n(k)]$ is a function of the form $x_1^{d_1}(k), x_2^{d_1}(k), \ldots, x_n^{d_n}(k)$, where d_i, $i \in \{1, 2, \ldots, n\}$ is a non-negative integer. The degree of a monomial is defined as $d = \sum_{i=1}^{n} d_i$. $\mathbf{p}(\mathbf{x}(k))$ is a polynomial if it can be expressed as a finite linear combination of monomials with real coefficients. $\mathbf{p}(\mathbf{x}(k)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(k))^2$ indicates the polynomial $\mathbf{p}(\mathbf{x}(k))$ is an SOS implying $\mathbf{p}(\mathbf{x}(k)) \ge 0$ where $\mathbf{q}_j(\mathbf{x}(k))$ is a polynomial and m is a non-zero positive integer. $\mathbf{Z} \succ 0$, $\mathbf{Z} \succcurlyeq 0$, $\mathbf{Z} \prec 0$ and $\mathbf{Z} \preccurlyeq 0$ mean that all the elements of the matrix $\mathbf{Z}$ are positive, semipositive, negative and seminegative, respectively. $\mathbf{R}^{\mathrm{T}}$ stands for the transpose of a real matrix $\mathbf{R}$. $\underline{n} = \{1, 2, \ldots, n\}$ where $n \in \mathbb{Z}^+$ denotes the order of fuzzy model. $\underline{p} = \{1, 2, \ldots, p\}$ where $p \in \mathbb{Z}^+$ is the number of rules of the fuzzy model. $\underline{c} = \{1, 2, \ldots, c\}$ where $c \in \mathbb{Z}^+$ is the rule number of fuzzy controller. $\underline{d} = \{1, 2, \ldots, d\}$, where $d \in \mathbb{Z}^+$ is the time delay. We denote the normalised grades of membership of fuzzy model and fuzzy controller as $\mathbf{w}(\mathbf{x}(k)) = [w_1(\mathbf{x}(k)), w_2(\mathbf{x}(k)), \ldots, w_i(\mathbf{x}(k))]$, $i \in \underline{p}$ and $\mathbf{m}(\mathbf{x}(k)) = [m_1(\mathbf{x}(k)), m_2(\mathbf{x}(k)), \ldots, m_j(\mathbf{x}(k))]$, $j \in \underline{c}$, respectively.

2.2 Discrete time polynomial fuzzy models with time delay

The dynamics of the discrete-time non-linear plant is described by p-rule polynomial fuzzy model. The ith rule subsystem is shown as below format:

$$\begin{aligned} &\text{Rule } i: \quad \text{IF } f_1(\mathbf{x}(k)) \text{ is } M_1^i \text{ AND} \cdots \text{AND } f_\Psi(\mathbf{x}(k)) \text{ is } M_\Psi^i \\ &\text{THEN } \mathbf{x}(k+1) = \mathbf{A}_{i0}(\mathbf{x}(k))\mathbf{x}(k) \\ &\qquad + \sum_{l=1}^{d} \mathbf{A}_{il}\mathbf{x}(k - \tau_l) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{u}(k) \end{aligned} \tag{1}$$

$$\mathbf{x}(k) = \boldsymbol{\phi}(k), \quad k = [-\tau_{\max}, 0], \tag{2}$$

where M_l^i, $l = \{1, 2, \ldots, \Psi\}$ is the fuzzy set in rule i corresponding to the premise variable $f_l(\mathbf{x}(k))$, $l = \{1, 2, \ldots, \Psi\}$; Ψ is a positive integer; $\boldsymbol{\phi}(k)$ is the vector valued initial function; $\mathbf{A}_{i0}(\mathbf{x}(k)) \in \Re^{n \times n}$, $\mathbf{A}_{il} \in \Re^{n \times n}$ and $\mathbf{B}_i(\mathbf{x}(k)) \in \Re^{n \times m}$ are known polynomial system, time delay and input matrices, respectively; $\mathbf{x}(k) \in \Re^n$ and $\mathbf{u}(k) \in \Re^m$ are state vector and control input vector, respectively; τ_l, $l \in \underline{d} = \{1, 2, \ldots, d\}$, is the constant time delay. Therefore, the system dynamics is described as below format:

$$\begin{aligned} \mathbf{x}(k+1) = &\sum_{i=1}^{p} w_i(\mathbf{x}(k))\Big(\mathbf{A}_{i0}(\mathbf{x}(k))\mathbf{x}(k) \\ &+ \sum_{l=1}^{d} \mathbf{A}_{il}\mathbf{x}(k - \tau_l) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{u}(k)\Big), \end{aligned} \tag{3}$$

where $w_i(\mathbf{x}(k))$ is the normalised grade of membership and satisfies

$$w_i(\mathbf{x}(k)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(k)))}{\sum_{k=1}^{p} \prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{x}(k)))} \ge 0 \tag{4}$$

$$\sum_{i=1}^{p} w_i(\mathbf{x}(k)) = 1 \tag{5}$$

$\mu_{M_l^i}(f_l(\mathbf{x}(k)))$ is the grade of membership corresponding to the fuzzy term M_l^i.

Definition 1: A system is said to be positive if, given the non-negative initial condition $\boldsymbol{\phi}(\cdot) \geqslant 0$, the corresponding trajectory $\mathbf{x}(k)$ remains in the positive orthant for all $k \geqslant 0$ [28].

Lemma 1: The discrete time polynomial fuzzy model based system with time delay (3) with $\mathbf{u}(k) = \mathbf{0}$ is said to be positive if the system and time delay matrices satisfy $\mathbf{A}_{i0}(\mathbf{x}(k)) \geqslant 0$ and $\mathbf{A}_{il} \geqslant 0$ [28].

2.3 Polynomial fuzzy controller

The jth rule of the PFMB controller using the IPC design concept is described as below:

$$\begin{aligned}&\text{Rule } j\text{:} \quad \text{IF } g_1(\mathbf{x}(k)) \text{ is } N_1^j \text{ AND} \cdots \text{AND } g_\Omega(\mathbf{x}(k)) \text{ is } N_\Omega^j \\ &\text{THEN } \mathbf{u}(k) = \mathbf{G}_j(\mathbf{x}(k))\mathbf{x}(k),\end{aligned} \tag{6}$$

where $g_l(\mathbf{x}(k))$, $l \in \{1, 2, \ldots, \Omega\}$, is the premise variable corresponding to fuzzy set N_l^j in rule j and Ω is a positive integer, $\mathbf{G}_j(\mathbf{x}(k)) \in \Re^{m \times n}$, $j \in \underline{c}$, is the polynomial feedback gain to be determined. The polynomial fuzzy controller is described as

$$\mathbf{u}(k) = \sum_{j=1}^{c} m_j(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))\mathbf{x}(k), \tag{7}$$

where $m_j(\mathbf{x}(k))$ is the normalised grade of membership satisfies

$$m_j(\mathbf{x}(k)) = \frac{\prod_{l=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(k)))}{\sum_{k=1}^{c} \prod_{l=1}^{\Omega} \mu_{N_l^k}(g_l(\mathbf{x}(k)))} \geq 0 \tag{8}$$

$$\sum_{j=1}^{c} m_j(\mathbf{x}(k)) = 1 \tag{9}$$

$\mu_{N_l^j}(g_l(\mathbf{x}(k)))$ is the grade of membership corresponding to the fuzzy term N_l^i.

Remark 1: In the IPC design concept [24, 25, 32], we let the membership function $m_j(\mathbf{x}(k))$ of the PFMB controller to be freely chosen as $m_j(\mathbf{x}(k)) \neq w_i(\mathbf{x}(k))$ for any $p \neq c$. Furthermore, we can also choose number of rules of the polynomial fuzzy controller which is same as polynomial fuzzy model (i.e. $p = c$) but membership functions of the polynomial fuzzy controller are freely chosen (i.e. $m_j(\mathbf{x}(k)) \neq w_i(\mathbf{x}(k))$) independently of the polynomial fuzzy model. No matter what kind of membership functions are chosen for fuzzy model and fuzzy controller, the membership functions need to satisfy the property (4)–(5) and (8)–(9).

2.4 Discrete time polynomial fuzzy models based control system with time delay

Formed by polynomial fuzzy model (3) and controller (7), the discrete time PFMB control system with time delay is

$$\begin{aligned}\mathbf{x}(k+1) = {} & \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}(k)) m_j(\mathbf{x}(k)) \Big((\mathbf{A}_{i0}(\mathbf{x}(k)) \\ & + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)))\mathbf{x}(k) + \sum_{l=1}^{d} \mathbf{A}_{il}\mathbf{x}(k - \tau_l) \Big).\end{aligned} \tag{10}$$

Lemma 2: The discrete time PFMB system with time delay (10) is controlled positive for $\boldsymbol{\phi}(\cdot) \geqslant 0$ if $\mathbf{A}_{i0}(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)) \geqslant 0$ and $\mathbf{A}_{il} \geqslant 0$ [28].

3 Positivity and stability analysis

To obtain a favourable formulation which is independent of time delay, we employ the dual system of the original system [28, 36] for the stability analysis. In a dual system, the matrices are transposed of the ones in the original system. Dual system of (10) is described as below:

$$\begin{aligned}\mathbf{x}(k+1) = {} & \sum_{i=1}^{p} \sum_{j=1}^{p} w_i(\mathbf{x}(k)) m_j(\mathbf{x}(k)) \Big((\mathbf{A}_{i0}(\mathbf{x}(k)) \\ & + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))^{\mathrm{T}}\mathbf{x}(k) + \sum_{l=1}^{d} \mathbf{A}_{il}^{\mathrm{T}}\mathbf{x}(k - \tau_l) \Big).\end{aligned} \tag{11}$$

In the following, we prove the equivalence of stability between two systems under duality principals.

Proof: Consider the discrete time PFMB control system (10)

$$\begin{aligned}\mathbf{x}(k+1) = {} & \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}(k)) m_j(\mathbf{x}(k)) \Big((\mathbf{A}_{i0}(\mathbf{x}(k)) \\ & + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)))\mathbf{x}(k) + \sum_{l=1}^{d} \mathbf{A}_{il}\mathbf{x}(k - \tau_l) \Big) \\ = {} & \sum_{i_k=1}^{p} \sum_{j_k=1}^{c} \sum_{l_k=1}^{d} \sum_{i_{k-1}=1}^{p} \sum_{j_{k-1}=1}^{c} \sum_{l_{k-1}=1}^{d} \cdots \\ & \times \sum_{i_0=1}^{p} \sum_{j_0=1}^{c} \sum_{l_0=1}^{d} w_i(\mathbf{x}(k)) m_j(\mathbf{x}(k)) \\ & \times w_{i_{k-1}}(\mathbf{x}(k-1)) m_{j_{k-1}}(\mathbf{x}(k-1)) \cdots \\ & \times w_{i_0}(\mathbf{x}(0)) m_{j_0}(\mathbf{x}(0)) \\ & \times ((\mathbf{A}_{i0}(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{A}_{il}) \\ & \times (\mathbf{A}_{i_{k-1}0}(\mathbf{x}(k-1)) + \mathbf{B}_{i_{k-1}}(\mathbf{x}(k-1)) \\ & \times \mathbf{G}_{j_{k-1}}(\mathbf{x}(k-1)) + \mathbf{A}_{i_{k-1}l_{k-1}}) \cdots \\ & \times (\mathbf{A}_{i_00}(\mathbf{x}(0)) + \mathbf{B}_{i_0}(\mathbf{x}(0))\mathbf{G}_{j_0}(\mathbf{x}(0)) + \mathbf{A}_{i_0l_0}) \\ & \times \mathbf{x}(-\tau_l - \tau_{l_{k-1}} - \cdots \tau_{l_0})).\end{aligned} \tag{12}$$

From (11), we can get the following expression:

$$\begin{aligned}\mathbf{x}(k) = {} & \sum_{i=1}^{p} \sum_{j=1}^{p} w_i(\mathbf{x}(k-1)) m_j(\mathbf{x}(k-1)) \Big((\mathbf{A}_{i0}(\mathbf{x}(k-1)) \\ & + \mathbf{B}_i(\mathbf{x}(k-1))\mathbf{G}_j(\mathbf{x}(k-1))^{\mathrm{T}}\mathbf{x}(k-1) \\ & + \sum_{l=1}^{d} \mathbf{A}_{il}^{\mathrm{T}}\mathbf{x}(k - \tau_l - 1) \Big) \\ \mathbf{x}(k-1) = {} & \sum_{i=1}^{p} \sum_{j=1}^{p} w_i(\mathbf{x}(k-2)) m_j(\mathbf{x}(k-2)) \Big((\mathbf{A}_{i0}(\mathbf{x}(k-2)) \\ & + \mathbf{B}_i(\mathbf{x}(k-2))\mathbf{G}_j(\mathbf{x}(k-2))^{\mathrm{T}}\mathbf{x}(k-2) \\ & + \sum_{l=1}^{d} \mathbf{A}_{il}^{\mathrm{T}}\mathbf{x}(k - \tau_l - 2) \Big) \\ & \ldots\end{aligned} \tag{13}$$

Therefore, the dual system (11) can be rewritten as

$$
\begin{aligned}
\mathbf{x}(k+1) &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\Big((\mathbf{A}_{i0}(\mathbf{x}(k)) \\
&\quad +\mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))^{\mathrm{T}}\mathbf{x}(k) + \sum_{l=1}^{d}\mathbf{A}_{il}^{\mathrm{T}}\mathbf{x}(k-\tau_l)\Big) \\
&= \sum_{i_k=1}^{p}\sum_{j_k=1}^{c}\sum_{l_k=1}^{d}\sum_{i_{k-1}=1}^{p}\sum_{j_{k-1}=1}^{c}\sum_{l_{k-1}=1}^{d}\cdots \\
&\quad \times \sum_{i_0=1}^{p}\sum_{j_0=1}^{c}\sum_{l_0=1}^{d} w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) \\
&\quad \times w_{i_{k-1}}(\mathbf{x}(k-1))m_{j_{k-1}}(\mathbf{x}(k-1))\cdots \\
&\quad \times w_{i_0}(\mathbf{x}(0))m_{j_0}(\mathbf{x}(0)) \\
&\quad \times ((\mathbf{A}_{i0}(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{A}_{il})^{\mathrm{T}} \\
&\quad \times (\mathbf{A}_{i_{k-1}0}(\mathbf{x}(k-1)) + \mathbf{B}_{i_{k-1}}(\mathbf{x}(k-1)) \\
&\quad \times \mathbf{G}_{j_{k-1}}(\mathbf{x}(k-1)) + \mathbf{A}_{i_{k-1}l_{k-1}})^{\mathrm{T}}\cdots \\
&\quad \times (\mathbf{A}_{i_00}(\mathbf{x}(0)) + \mathbf{B}_{i_0}(\mathbf{x}(0))\mathbf{G}_{j_0}(\mathbf{x}(0)) + \mathbf{A}_{i_0l_0})^{\mathrm{T}} \\
&\quad \times \mathbf{x}(-\tau_l - \tau_{l_{k-1}} - \cdots\tau_{l_0})\big).
\end{aligned} \tag{14}
$$

Based on the definition of Lyapunov stability theory, if we say that the dual system (14) is stable with initial condition $\mathbf{x}(0) = \boldsymbol{\phi}(\mathbf{0}) \ge 0$ and $\mathbf{x}(-\tau_l - \tau_{l_{k-1}} - \cdots\tau_{l_0}) = \boldsymbol{\phi}(-\tau_{\mathbf{l}} - \tau_{\mathbf{l_{k-1}}} - \cdots\tau_{\mathbf{l_0}}) \ge 0$, then we have $\mathbf{x}(k+1) \to 0$ as $k \to \infty$. Therefore, we get if $k \to \infty$, the following formulations is satisfied:

$$
\begin{aligned}
&((\mathbf{A}_{i0}(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{A}_{il})^{\mathrm{T}} \\
&\quad \times (\mathbf{A}_{i_{k-1}0}(\mathbf{x}(k-1)) + \mathbf{B}_{i_{k-1}}(\mathbf{x}(k-1)) \\
&\quad \times \mathbf{G}_{j_{k-1}}(\mathbf{x}(k-1)) + \mathbf{A}_{i_{k-1}l_{k-1}})^{\mathrm{T}}\cdots \\
&\quad \times (\mathbf{A}_{i_00}(\mathbf{x}(0)) + \mathbf{B}_{i_0}(\mathbf{x}(0))\mathbf{G}_{j_0}(\mathbf{x}(0)) + \mathbf{A}_{i_0l_0})^{\mathrm{T}}) \to 0. \\
&= ((\mathbf{A}_{i_00}(\mathbf{x}(0)) + \mathbf{B}_{i_0}(\mathbf{x}(0))\mathbf{G}_{j_0}(\mathbf{x}(0)) + \mathbf{A}_{i_0l_0})\cdots \\
&\quad \times (\mathbf{A}_{i_{k-1}0}(\mathbf{x}(k-1)) + \mathbf{B}_{i_{k-1}}(\mathbf{x}(k-1)) \\
&\quad \times \mathbf{G}_{j_{k-1}}(\mathbf{x}(k-1)) + \mathbf{A}_{i_{k-1}l_{k-1}}) \\
&\quad \times (\mathbf{A}_{i0}(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{A}_{il}))^{\mathrm{T}} \to 0,
\end{aligned} \tag{15}
$$

and if formulation (15) is satisfied, we obviously have

$$
\begin{aligned}
&((\mathbf{A}_{i_00}(\mathbf{x}(0)) + \mathbf{B}_{i_0}(\mathbf{x}(0))\mathbf{G}_{j_0}(\mathbf{x}(0)) + \mathbf{A}_{i_0l_0})\cdots \\
&\quad \times (\mathbf{A}_{i_{k-1}0}(\mathbf{x}(k-1)) + \mathbf{B}_{i_{k-1}}(\mathbf{x}(k-1)) \\
&\quad \times \mathbf{G}_{j_{k-1}}(\mathbf{x}(k-1)) + \mathbf{A}_{i_{k-1}l_{k-1}}) \\
&\quad \times (\mathbf{A}_{i0}(\mathbf{x}(k)) + \mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)) + \mathbf{A}_{il})) \to 0.
\end{aligned} \tag{16}
$$

From (16) and (12), we can obtain the stability of dual system (11) which is equivalent to (10) in terms of stability. Therefore, we can guarantee the PFMB controller (7) based on IPC design concept is able to stabilise the original system (10) and its dual system (16) at the same time. This completes proof. □

3.1 SOS-based positivity analysis

First, we investigate the positivity analysis for discrete-time PFMB control systems with time delay, i.e. we guarantee trajectory $\mathbf{x}(k) \succcurlyeq 0$ if the initial condition $\boldsymbol{\phi}(\cdot) \succcurlyeq 0$. Using Lemma 2, SOS-based positivity conditions are described in Theorem 1.

Theorem 1: The discrete time PFMB control system with time delay (10) with the initial condition $\boldsymbol{\phi}(\cdot) \succcurlyeq 0$ is controlled positive if there exist $\lambda \in \Re^n$ and $\mathbf{y}_s^j(\mathbf{x}(k)) \in \Re^m$ for $j \in \underline{c}$ and $s \in \underline{n}$ such that the following SOS-based conditions are satisfied:

$$a_{fs}^{i0}(\mathbf{x}(k))\lambda_s + \mathbf{b}_f^i(\mathbf{x}(k))\mathbf{y}_s^j(\mathbf{x}(k)) \text{ is SOS},\ i \in \underline{p};\ j \in \underline{c}; f, s \in \underline{n} \tag{17}$$

$$a_{fs}^{il} \text{ is SOS},\ i \in \underline{p};\ l \in \underline{d}; f, s \in \underline{n}, \tag{18}$$

where $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]^{\mathrm{T}} \succ 0$, $a_{fs}^{i0}(\mathbf{x}(k))$ and a_{fs}^{il} are the (f,s)-th element of the system and time delay matrices $\mathbf{A}_{i0}(\mathbf{x}(k))$ and $\mathbf{A}_{il}$, respectively; $\mathbf{B}_i(\mathbf{x}(k)) = [\mathbf{b}_1^i(\mathbf{x}(k))^{\mathrm{T}}, \mathbf{b}_2^i(\mathbf{x}(k))^{\mathrm{T}}, \cdots, \mathbf{b}_n^i(\mathbf{x}(k))^{\mathrm{T}}]^{\mathrm{T}}$, $i \in \underline{p}$, $f, s \in \underline{n}$; the polynomial fuzzy controller is $\mathbf{G}_j(\mathbf{x}(k)) = \left[\frac{\mathbf{y}_1^j(\mathbf{x}(k))}{\lambda_1}, \frac{\mathbf{y}_2^j(\mathbf{x}(k))}{\lambda_2}, \cdots, \frac{\mathbf{y}_n^j(\mathbf{x}(k))}{\lambda_n}\right]$ where $\mathbf{y}_1^j(\mathbf{x}(k))$, $\mathbf{y}_2^j(\mathbf{x}(k))$, …, $\mathbf{y}_n^j(\mathbf{x}(k)) \in \Re^m$ for $j \in \underline{c}$ are to be determined.

Proof: The obtained SOS-based positivity conditions (17) and (18) can be realised based on Lemma 2. □

3.2 SOS-based stability analysis

Subject to positivity conditions based on Theorem 1, the following polynomial Lyapunov functional candidate is employed to investigate stability of (10) and (11)

$$V(\mathbf{x}(k)) = \mathbf{x}^{\mathrm{T}}(k)\lambda + \sum_{m=1}^{p}\sum_{l=1}^{d}\sum_{q=1}^{\tau_d}\left(\mathbf{x}(k-q)^{\mathrm{T}}\mathbf{A}_{ml}\lambda\right), \tag{19}$$

where $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]^{\mathrm{T}} \succ 0$.

From (11) and (19), we get

$$
\begin{aligned}
\triangle V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\Big[\mathbf{x}^{\mathrm{T}}(k)(\mathbf{A}_{i0}(\mathbf{x}(k)) \\
&\quad +\mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k))) + \mathbf{x}^{\mathrm{T}}(k-\tau_l)\sum_{l=1}^{d}\mathbf{A}_{il}\Big]\lambda \\
&\quad + \sum_{m=1}^{p}\sum_{l=1}^{d}\left(\mathbf{x}(k)^{\mathrm{T}}\mathbf{A}_{ml} - \mathbf{x}(k-\tau_l)^{\mathrm{T}}\mathbf{A}_{ml}\right)\lambda \\
&\quad - \mathbf{x}^{\mathrm{T}}(k)\lambda \\
&\le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))[\mathbf{x}^{\mathrm{T}}(k)(\mathbf{A}_{i0}(\mathbf{x}(k)) \\
&\quad +\mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)))]\lambda \\
&\quad + \sum_{i=1}^{p}\sum_{l=1}^{d}\mathbf{x}^{\mathrm{T}}(k-\tau_l)\mathbf{A}_{il}\lambda - \mathbf{x}^{\mathrm{T}}(k)\lambda \\
&\quad + \sum_{m=1}^{p}\sum_{l=1}^{d}\left(\mathbf{x}(k)^{\mathrm{T}}\mathbf{A}_{ml} - \mathbf{x}(k-\tau_l)^{\mathrm{T}}\mathbf{A}_{ml}\right)\lambda \\
&\le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))[\mathbf{x}^{\mathrm{T}}(k)(\mathbf{A}_{i0}(\mathbf{x}(k)) \\
&\quad +\mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)))]\lambda - \mathbf{x}^{\mathrm{T}}(k)\lambda \\
&\quad + \sum_{m=1}^{p}\sum_{l=1}^{d}\mathbf{x}(k)^{\mathrm{T}}\mathbf{A}_{ml}\lambda \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\mathbf{x}^{\mathrm{T}}(k)\Big[(\mathbf{A}_{i0}(\mathbf{x}(k)) \\
&\quad +\mathbf{B}_i(\mathbf{x}(k))\mathbf{G}_j(\mathbf{x}(k)))\lambda - \lambda + \sum_{m=1}^{p}\sum_{l=1}^{d}\mathbf{A}_{ml}\lambda\Big] \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\mathbf{x}^{\mathrm{T}}(k)\Big[(\mathbf{A}_{i0}(\mathbf{x}(k)) \\
&\quad + \sum_{m=1}^{p}\sum_{l=1}^{d}\mathbf{A}_{ml})\lambda + \mathbf{B}_i(\mathbf{x}(k))\sum_{s=1}^{n}\mathbf{y}_s^j(\mathbf{x}(k)) - \lambda\Big] \\
&= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}_{ij}(\mathbf{x}(k)),
\end{aligned} \tag{20}
$$

where

$$\mathbf{Q}_{ij}(\mathbf{x}(k)) = \left(\mathbf{A}_{i0}(\mathbf{x}(k)) + \sum_{m=1}^{p}\sum_{l=1}^{d}\mathbf{A}_{ml}\right)\lambda + \mathbf{B}_i(\mathbf{x}(k))\sum_{s=1}^{n}\mathbf{y}_s^j(\mathbf{x}(k)) - \lambda = \left[q_1^{ij}(\mathbf{x}(k)), q_2^{ij}(\mathbf{x}(k)), \ldots, q_n^{ij}(\mathbf{x}(k))\right]^{\mathrm{T}}. \tag{21}$$

Remark 2: Together with guaranteeing asymptotical positivity of (10) by satisfying conditions in Theorem 1, due to the stability equivalence of (10) and the dual system (11), the asymptotic stability of (10) is guaranteed by $V(k) > 0$ and $\triangle V(k) < 0$ (excluding $\mathbf{x}(k) = 0$) based on Lyapunov stability theory. This is realised using $q_s^{ij}(\mathbf{x}(k)) < 0$ for $i \in \underline{p}, j \in \underline{c}, s \in \underline{n}$.

3.3 Relaxed membership functions dependent stability analysis

Solving the positivity and asymptotic stability conditions for (10) based on Remark 2 may found very conservative since no information from membership functions have been introduced in the stability analysis. In this section, to reduce such conservativeness, we will introduce information from membership functions by considering boundary and relationship constraint between membership functions of the fuzzy model and controller. First, we consider the lower and upper boundary of membership functions as follows:

$$\underline{\eta}_i \le w_i(\mathbf{x}(k)) \le \overline{\eta}_i, \tag{22}$$

$$\underline{\varphi}_j \le m_j(\mathbf{x}(k)) \le \overline{\varphi}_j, \tag{23}$$

$$\underline{\rho}_{ij} \le w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) \le \overline{\rho}_{ij}, \tag{24}$$

where $\underline{\eta}_i$ and $\overline{\eta}_i$ are the lower and upper bounds of polynomial fuzzy model membership functions, respectively, $\underline{\varphi}_j$ and $\overline{\varphi}_j$ are the lower and upper bounds of fuzzy controller membership functions, respectively, $\underline{\rho}_{ij}$ and $\overline{\rho}_{ij}$ are the lower and upper bounds of $w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))$, respectively.

We then consider the relationship constraint information between membership functions of the fuzzy model and the controller

$$\sum_{i=1}^{p}\sum_{j=1}^{c}(\sigma_{rij}w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) + \ell_{ri}w_i(\mathbf{x}(t)) + \Gamma_{rj}m_j(\mathbf{x}(k))) - \varphi_r \ge 0, \tag{25}$$

where σ_{rij}, ℓ_{ri}, Γ_{rj} and φ_r are the predefined parameters which satisfy the formulation (25). As can be seen from the formulation (25), it gives a general constraint relationship information between membership functions of fuzzy model $w_i(\mathbf{x}(k))$ and fuzzy controller $m_j(\mathbf{x}(k))$. Take the example (22) that membership function of fuzzy model corresponding to fuzzy rule 1 satisfy $\underline{\eta}_1 \le w_1(\mathbf{x}(k))$, we can treat it as a special case when $\sigma_{rij} = \Gamma_{rj} = 0$ for $i \in \underline{p}, j \in \underline{c}$, $\ell_{r1} = 1, \ell_{ri} = 0, \varphi_r = \underline{\eta}_1$ for $i \in 2, \ldots, p$. Therefore, using relationship constraint information (25), more information of membership functions can be included in stability and positivity analysis. We will then introduce the terms in (22)–(25) by slack matrices in following inequalities:

$$\sum_{i=1}^{p}(w_i(\mathbf{x}(k)) - \underline{\eta}_i)\mathbf{M}_i(\mathbf{x}(k)) \succcurlyeq 0, \tag{26}$$

$$\sum_{i=1}^{p}(\overline{\eta}_i - w_i(\mathbf{x}(k)))\mathbf{W}_i(\mathbf{x}(k)) \succcurlyeq 0, \tag{27}$$

$$\sum_{j=c}^{p}(m_j(\mathbf{x}(k)) - \underline{\varphi}_j)\mathbf{P}_j(\mathbf{x}(k)) \succcurlyeq 0, \tag{28}$$

$$\sum_{j=1}^{c}(\overline{\varphi}_j - m_j(\mathbf{x}(k)))\mathbf{Z}_j(\mathbf{x}(k)) \succcurlyeq 0, \tag{29}$$

$$\sum_{i=1}^{p}\sum_{j=1}^{c}(w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) - \underline{\rho}_{ij})\mathbf{H}_{ij}(\mathbf{x}(k)) \succcurlyeq 0, \tag{30}$$

$$\sum_{i=1}^{p}\sum_{j=1}^{c}(\overline{\rho}_{ij} - w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)))\mathbf{J}_{ij}(\mathbf{x}(k)) \succcurlyeq 0, \tag{31}$$

$$\sum_{r=1}^{R}\left(\sum_{i=1}^{p}\sum_{j=1}^{c}(\sigma_{rij}w_i(\mathbf{x}(t))m_j(\mathbf{x}(t)) + \ell_{ri}w_i(\mathbf{x}(t)) + \Gamma_{rj}m_j(\mathbf{x}(t))) - \varphi_r\right)\mathbf{T}_r(\mathbf{x}(t)) \succcurlyeq 0, \tag{32}$$

where $0 \preccurlyeq \mathbf{M}_i(\mathbf{x}(k)) \in \mathfrak{R}^n$, $0 \preccurlyeq \mathbf{W}_i(\mathbf{x}(k)) \in \mathfrak{R}^n$, $0 \preccurlyeq \mathbf{P}_j(\mathbf{x}(k)) \in \mathfrak{R}^n$, $0 \preccurlyeq \mathbf{Z}_j(\mathbf{x}(k)) \in \mathfrak{R}^n$, $0 \preccurlyeq \mathbf{H}_{ij}(\mathbf{x}(k)) \in \mathfrak{R}^n$, $0 \preccurlyeq \mathbf{J}_{ij}(\mathbf{x}(k)) \in \mathfrak{R}^n$ and $0 \preccurlyeq \mathbf{T}_r(\mathbf{x}(k)) \in \mathfrak{R}^n$ are polynomial vectors. $R \in \mathbb{Z}^+$ denotes the number of relationship constraint information of membership functions between polynomial fuzzy model and controllers.

From (26)–(32) and (20), we obtain

$$\begin{aligned}\triangle V(\mathbf{x}(k)) \le \mathbf{x}^{\mathrm{T}}(k)\sum_{i=1}^{p}\sum_{j=1}^{c}w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\mathbf{Q}_{ij}(\mathbf{x}(k)) &+ \sum_{i=1}^{p}(w_i(\mathbf{x}(k)) - \underline{\eta}_i)\mathbf{M}_i(\mathbf{x}(k)) \\ &+ \sum_{i=1}^{p}(\overline{\eta}_i - w_i(\mathbf{x}(k)))\mathbf{W}_i(\mathbf{x}(k)) \\ &+ \sum_{j=1}^{c}(m_j(\mathbf{x}(k)) - \underline{\varphi}_j)\mathbf{P}_j(\mathbf{x}(k)) \\ &+ \sum_{j=1}^{c}(\overline{\varphi}_j - m_j(\mathbf{x}(k)))\mathbf{Z}_j(\mathbf{x}(k)) \\ &+ \sum_{i=1}^{p}\sum_{j=1}^{c}(w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) - \underline{\rho}_{ij})\mathbf{H}_{ij}(\mathbf{x}(k)) \\ &+ \sum_{i=1}^{p}\sum_{j=1}^{c}(\overline{\rho}_{ij} - w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)))\mathbf{J}_{ij}(\mathbf{x}(k)) \\ &+ \sum_{r=1}^{R}\left(\sum_{i=1}^{p}\sum_{j=1}^{c}(\sigma_{rij}w_i(\mathbf{x}(k))m_j(\mathbf{x}(k)) + \ell_{ri}w_i(\mathbf{x}(k)) + \Gamma_{rj}m_j(\mathbf{x}(k))) - \varphi_r\right)\mathbf{T}_r(\mathbf{x}(k))\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{x}^{\mathrm{T}}(k)\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(k))m_j(\mathbf{x}(k))\Big(\mathbf{Q}_{ij}(\mathbf{x}(k)) \\
&\quad +\mathbf{M}_i(\mathbf{x}(k)) - \mathbf{W}_i(\mathbf{x}(k)) + \mathbf{P}_j(\mathbf{x}(k)) \\
&\quad -\mathbf{Z}_j(\mathbf{x}(k)) + \mathbf{H}_{ij}(\mathbf{x}(k)) - \mathbf{J}_{ij}(\mathbf{x}(k)) \\
&\quad +\sum_{r=1}^{R}(\sigma_{rij} + \ell_{ri} + \Gamma_{rj})\mathbf{T}_r(\mathbf{x}(k))\Big) \\
&\quad +\Big(\sum_{i=1}^{p}\big(\overline{\eta}_i\mathbf{W}_i(\mathbf{x}(k)) - \underline{\eta}_i\mathbf{M}_i(\mathbf{x}(k))\big) \\
&\quad +\sum_{j=1}^{c}\big(\overline{\varphi}_j\mathbf{Z}_j(\mathbf{x}(k)) - \underline{\varphi}_j\mathbf{P}_j(\mathbf{x}(k))\big) \\
&\quad +\sum_{i=1}^{p}\sum_{j=1}^{c}\big(\overline{\rho}_{ij}\mathbf{J}_{ij}(\mathbf{x}(k)) - \underline{\rho}_{ij}\mathbf{H}_{ij}(\mathbf{x}(k))\big) \\
&\quad +\sum_{r=1}^{R}\varphi_r\mathbf{T}_r(\mathbf{x}(k))\Big).
\end{aligned} \tag{33}
$$

We define $\mathbf{U}_{ij}(\mathbf{x}(k)) \geqslant \mathbf{Q}_{ij}(\mathbf{x}(k)) + \mathbf{M}_i(\mathbf{x}(k)) - \mathbf{W}_i(\mathbf{x}(k)) + \mathbf{P}_j(\mathbf{x}(k)) - \mathbf{Z}_j(\mathbf{x}(k)) + \mathbf{H}_{ij}(\mathbf{x}(k)) - \mathbf{J}_{ij}(\mathbf{x}(k)) + \sum_{r=1}^{R}(\sigma_{rij} + \ell_{kri} + \Gamma_{rj})\mathbf{T}_r(\mathbf{x}(k))$ and $\mathbf{U}_{ij} \geqslant 0$. Therefore, we have

$$
\begin{aligned}
\triangle V(\mathbf{x}(k)) &\le \mathbf{x}^{\mathrm{T}}(k)\sum_{i=1}^{p}\sum_{j=1}^{c}\overline{\rho}_{ij}\mathbf{U}_{ij}(\mathbf{x}(k)) \\
&\quad +\Big(\sum_{i=1}^{p}(\overline{\eta}_i\mathbf{W}_i(\mathbf{x}(k)) - \underline{\eta}_i\mathbf{M}_i(\mathbf{x}(k))) \\
&\quad +\sum_{j=1}^{c}(\overline{\varphi}_j\mathbf{Z}_j(\mathbf{x}(k)) - \underline{\varphi}_j\mathbf{P}_j(\mathbf{x}(k))) \\
&\quad +\sum_{i=1}^{p}\sum_{j=1}^{c}(\overline{\rho}_{ij}\mathbf{J}_{ij}(\mathbf{x}(k)) - \underline{\rho}_{ij}\mathbf{H}_{ij}(\mathbf{x}(k))) \\
&\quad +\sum_{r=1}^{R}\varphi_r\mathbf{T}_r(\mathbf{x}(k))\Big) \\
&= \mathbf{x}^{\mathrm{T}}(k)\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\overline{\rho}_{ij}\mathbf{U}_{ij}(\mathbf{x}(k)) + \overline{\eta}_i\mathbf{W}_i(\mathbf{x}(k)) \\
&\quad -\underline{\eta}_i\mathbf{M}_i(\mathbf{x}(k)) + \overline{\varphi}_j\mathbf{Z}_j(\mathbf{x}(k)) - \underline{\varphi}_j\mathbf{P}_j(\mathbf{x}(k)) \\
&\quad +\overline{\rho}_{ij}\mathbf{J}_{ij}(\mathbf{x}(k)) - \underline{\rho}_{ij}\mathbf{H}_{ij}(\mathbf{x}(k)) + \sum_{r=1}^{R}\varphi_r\mathbf{T}_r(\mathbf{x}(k))\Big).
\end{aligned} \tag{34}
$$

From (34), the membership-function dependent asymptotic stability and positivity conditions of (10) subject to Theorem 1 are summarised in the following theorem.

Theorem 2: The PFMB control system (10) is positive and asymptotically stable if Theorem 1 and the following SOS-based conditions are satisfied:

$$
\lambda_s - \varepsilon_1 \text{ is SOS}, s \in \underline{n}, \tag{35}
$$

$$
\begin{aligned}
&-\Big(\sum_{i=1}^{p}\sum_{j=1}^{c}\big(\overline{\rho}_{ij}u_s^{ij}(\mathbf{x}(k)) + \overline{\eta}_i w_s^i(\mathbf{x}(k)) - \underline{\eta}_i m_s^i(\mathbf{x}(k)) \\
&+\overline{\varphi}_j z_s^j(\mathbf{x}(k)) - \underline{\varphi}_j p_s^j(\mathbf{x}(k)) + \overline{\rho}_{ij} j_s^{ij}(\mathbf{x}(k)) \\
&-\underline{\rho}_{ij} h_s^{ij}(\mathbf{x}(k)) + \sum_{r=1}^{R}\varphi_r t_s^r(\mathbf{x}(k))\big) + \varepsilon_2(\mathbf{x}(k))\Big) \\
&\text{is SOS}, i \in \underline{p}, j \in \underline{c}, r \in \underline{R}, s \in \underline{n},
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
&u_s^{ij}(\mathbf{x}(k)) - \Big(q_s^{ij}(\mathbf{x}(k)) + m_s^i(\mathbf{x}(k)) - w_s^i(\mathbf{x}(k)) \\
&\quad + p_s^j(\mathbf{x}(k)) - z_s^j(\mathbf{x}(k)) + h_s^{ij}(\mathbf{x}(k)) - j_s^{ij}(\mathbf{x}(k)) \\
&\quad + \sum_{r=1}^{R}(\sigma_{rij} + \ell_{kri} + \Gamma_{rj})t_s^r(\mathbf{x}(k))\Big) \\
&\quad \text{is SOS}, \quad i \in \underline{p}, j \in \underline{c}, r \in \underline{R}, s \in \underline{n},
\end{aligned} \tag{37}
$$

$$
u_s^{ij}(\mathbf{x}(k)) \text{ is SOS}, \quad i \in \underline{p}, j \in \underline{c}, s \in \underline{n}, \tag{38}
$$

$$
w_s^i(\mathbf{x}(k))(\mathbf{x}(k)) \text{ is SOS}, \quad i \in \underline{p}, s \in \underline{n}, \tag{39}
$$

$$
m_s^i(\mathbf{x}(k)) \text{ is SOS}, \quad i \in \underline{p}, s \in \underline{n}, \tag{40}
$$

$$
z_s^j(\mathbf{x}(k)) \text{ is SOS}, \quad j \in \underline{c}, s \in \underline{n}, \tag{41}
$$

$$
p_s^j(\mathbf{x}(k))(\mathbf{x}(k)) \text{ is SOS}, \quad j \in \underline{c}, s \in \underline{n}, \tag{42}
$$

$$
j_s^{ij}(\mathbf{x}(k)) \text{ is SOS}, \quad i \in \underline{p}, j \in \underline{c}, s \in \underline{n}, \tag{43}
$$

$$
h_s^{ij}(\mathbf{x}(k)) \text{ is SOS}, \quad i \in \underline{p}, j \in \underline{c}, s \in \underline{n}, \tag{44}
$$

$$
t_s^r(\mathbf{x}(k))(\mathbf{x}(k)) \text{ is SOS}, \quad r \in \underline{R}, s \in \underline{n}, \tag{45}
$$

where $\varepsilon_1 > 0$ is a predefined scalar and $\varepsilon_2(\mathbf{x}(k)) > 0$ is a predefined scalar polynomial; $q_s^{ij}(\mathbf{x}(k))$ is defined in (21); the feedback gains and the other variables are defined in Theorem 1. λ is a decision variable vector and obtained by satisfying Theorems 1 and 2 via the SOS approach.

Proof: The SOS-based stability conditions (35)–(45) is obtained based on Lyapunov stability theory which has been carried out in the terms from (19)–(34). □

Remark 3: In terms of designing the slack matrices $\mathbf{M}_i(\mathbf{x}(k)) \in \mathfrak{R}^n$, $\mathbf{W}_i(\mathbf{x}(k)) \in \mathfrak{R}^n$, $\mathbf{P}_j(\mathbf{x}(k)) \in \mathfrak{R}^n$, $\mathbf{Z}_j(\mathbf{x}(k)) \in \mathfrak{R}^n$, $\mathbf{H}_{ij}(\mathbf{x}(k)) \in \mathfrak{R}^n$, $\mathbf{J}_{ij}(\mathbf{x}(k)) \in \mathfrak{R}^n$ and $\mathbf{T}_r(\mathbf{x}(k)) \in \mathfrak{R}^n$ in SOS-based stability conditions (36)–(45), we only predefine the polynomial of degree in $\mathbf{x}(k)$ of these slack matrices, but the specific parameters corresponding to $\mathbf{x}(k)$ will be numerically solved by MATLAB third party toolbox SOSTOOLS. In terms of parameters $\underline{\eta}_i$, $\overline{\eta}_i$, $\underline{\varphi}_j$, $\overline{\varphi}_j$, $\underline{\rho}_{ij}$, $\overline{\rho}_{ij}$, ℓ_{ri}, Γ_{rj} and φ_r in SOS-based stability conditions (36)–(37), they can be obtained numerically by satisfying (22)–(25).

4 Simulation example

In this section, we employ a numerical example of discrete-time PFMB with time delay to validate the proposed theorems in the previous section. The exampled system has three fuzzy rules with the following subsystem, input and time-delay matrices:

$$\mathbf{x}(k) = [x_1(k) \quad x_2(k)]^{\mathrm{T}},$$
$$\mathbf{A}_{10}(x_1(k)) = \begin{bmatrix} 0.06b + 0.7 + 0.015x_1(k) - 0.001x_1(k)^2 & 0.2 \\ 0.2 & 0.3 \end{bmatrix},$$
$$\mathbf{A}_{20}(x_1(k)) = \begin{bmatrix} 0.4 & 0.1 - 0.01x_1(k) \\ 0.2 & 0.1a \end{bmatrix},$$
$$\mathbf{A}_{30}(x_1(k)) = \begin{bmatrix} 0.03 & 0.4 \\ 0.24 + 0.01x_1(k) & 0.06 + 0.0003x_1(k)^2 \end{bmatrix},$$
$$\mathbf{B}_1(x_1(k)) = \begin{bmatrix} 0.1b + 0.4 \\ 0.1 - 0.001x_1(k)^2 \end{bmatrix},$$
$$\mathbf{B}_2(x_1(k)) = \begin{bmatrix} 1 + 0.015x_1(k)^2 \\ -0.1 + 0.001x_1(k)^2 \end{bmatrix},$$
$$\mathbf{B}_3(x_1(k)) = \begin{bmatrix} -1 + 0.005x_1(k)^2 \\ 0.1 - 0.001x_1(k)^2 \end{bmatrix},$$
$$\mathbf{A}_{11} = \mathbf{A}_{21} = \mathbf{A}_{31} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\mathbf{A}_{12} = \mathbf{A}_{22} = \mathbf{A}_{32} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix},$$

where predefined constant scalars a and b are setting in the range of $5.5 \leq a \leq 9$ and $4.5 \leq b \leq 6.7$ with the interval of 0.5 and 0.2, respectively. In this way, a two-dimensional space is provided. When the positivity and stability conditions based on Remark 2 or Theorem 2 given in this paper satisfied, corresponding specific value of parameter a and b within the range will be marked with corresponding symbol. Otherwise, leave it blank. Therefore, the size of the stability region can be visualised. The feasible regions guaranteeing the positivity and stability of PFMB control system with time delay can be demonstrated and compared in one two-dimensional space.

The membership functions of three-rule polynomial fuzzy model are selected as

$$w_1(x_1) = 1 - \frac{1}{(1 + e^{-(x_1 - 6)})},$$
$$w_2(x_1) = 1 - w_1(x_1) - w_3(x_1),$$
$$w_3(x_1) = \frac{1}{(1 + e^{-(x_1 - 14)})}.$$

Based on IPC design method, the membership functions of a two-rule polynomial fuzzy controller which is designed to guarantee the positivity and stability of the system are defined as follows:

$$m_1(x_1) = e^{-(x_1 - 10)^2/12},$$
$$m_2(x_1) = \mu_{N_1^2}(x_1) = 1 - m_1(x_1).$$

The shape of the above membership functions is shown in Fig. 1.

We first use the asymptotic stability conditions in Remark 2 which considers no information of membership functions to design the PFMB controller and guarantee the system positivity and stability. If the term $\mathbf{y}_s^j(\mathbf{x}(k))$ is set as polynomial of degrees 0 to 4 in x_1 for $j \in \underline{c}, s \in \underline{n}$, no feasible solutions can be found to guarantee the stability and positivity of discrete-time PFMB control system with time delay. Therefore, we need a relaxed formulation to find the solution, and we employ Theorem 2 which considers the relationship constraint information between membership functions of polynomial fuzzy model and the controller along with the boundary information of the membership functions. The term $\mathbf{y}_s^j(\mathbf{x}(k))$ is set as polynomial of degrees 0 to 4 in x_1, $\mathbf{U}_{ij}(\mathbf{x}(k))$, $\mathbf{W}_i(\mathbf{x}(k))$, $\mathbf{M}_i(\mathbf{x}(k))$, $\mathbf{Z}_j(\mathbf{x}(k))$, $\mathbf{P}_j(\mathbf{x}(k))$, $\mathbf{J}_{ij}(\mathbf{x}(k))$ and $\mathbf{H}_{ij}(\mathbf{x}(k))$ all as polynomial of degree 0 in x_1 and $\mathbf{T}_r(\mathbf{x}(k))$ as polynomial of degrees 0 to 4 in x_1 for $i \in \underline{p}, j \in \underline{c}, s \in \underline{n}$. No feasible solution can be found for the system whose matrices

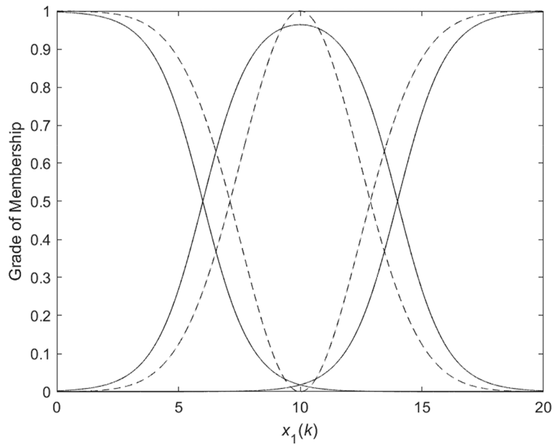

Fig. 1 *Membership functions of the polynomial fuzzy model (solid lines) and polynomial fuzzy controller (dotted lines)*

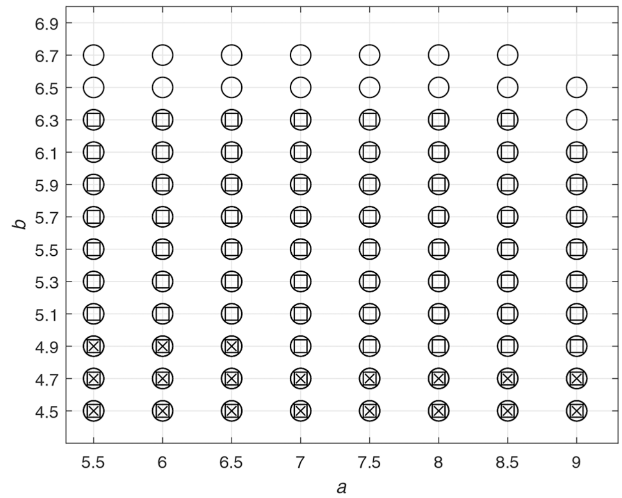

Fig. 2 *Feasible regions given by Theorem 2 with $R = 2$, $R = 3$ and $R = 4$ indicated by '×', '□' and '∘', respectively*

$\mathbf{T}_r(\mathbf{x}(k))$ with $R = 1$ (cf. (32)). Feasible solutions can be found for those with $R = 2$, $R = 3$ and $R = 4$ to guarantee the stability and positivity of discrete time PFMB control system with time delay as shown in Fig. 2 represented by symbols '+', '□' and '∘', respectively. The relationship constraint information of membership functions of the polynomial fuzzy model and controller, cf. (32), from $r = 1$ to $r = 4$ is shown in Table 1. The upper and lower boundary of membership functions, cf. from (26) to (31), is shown in Table 2.

If we compare the feasible regions based on Theorem 2 and those based on Remark 2 (see Fig. 2), we can see that the feasible regions based on Theorem 2 are larger than those based on Remark 2. This clearly shows that the positivity and stability conditions in Theorem 2 which considers the information of membership functions lead to more relaxed stability and positivity conditions when compared to the stability formulation in Remark 2 with no information of membership functions. The reason of obtained relaxed stability conditions is that the stability conditions obtained are not for any shape of membership functions but dedicated to the relationship constraint between the membership functions of the fuzzy model and the controller to be controlled, therefore the conservativeness of SOS based stability and positivity conditions is reduced. To demonstrate the effect of amount of the relationship constraint information from membership functions on stability and positivity analysis, we can see that the feasible regions based on Theorem 2 with $R = 4$ are the largest compared with those with $R = 2$ and $R = 3$. The feasible regions based on Theorem 2 with $R = 3$ are smaller than those with $R = 4$ but larger than those with $R = 2$ (see Fig. 2). There is no feasible regions found with $R = 1$. This shows introducing more relationship constraint information between the membership functions of fuzzy model and controller

Table 1 Relationship information between membership functions of fuzzy model and fuzzy controller

r	Parameters referring to (32)
1	$\sigma_{111} = 0.0143$, $\sigma_{112} = 0.1622$
	$\sigma_{121} = 0.1679, \sigma_{122} = 0.0530,$
	$\sigma_{131} = -0.1523$, $\sigma_{132} = -0.1187$
	$\ell_{11} = -0.1953$, $\ell_{12} = 0.1469$, $\ell_{13} = 0.1826$
	$\Gamma_{11} = -0.1695$, $\Gamma_{12} = 0.1361, \varphi_1 = 0.0934$
2	$\sigma_{211} = -0.2709$, $\sigma_{212} = -0.1181$
	$\sigma_{221} = -0.3730$, $\sigma_{222} = 0.0184$
	$\sigma_{231} = -0.1382$, $\sigma_{232} = -0.2729$
	$\ell_{21} = -0.1864$, $\ell_{22} = 0.0395$, $\ell_{23} = 0.3850$
	$\Gamma_{21} = 0.1504$, $\Gamma_{22} = 0.2168, \varphi_2 = -0.1859$
3	$\sigma_{311} = 0.0535$, $\sigma_{312} = 0.0213$
	$\sigma_{321} = -0.1592, \sigma_{322} = -0.1780$
	$\sigma_{331} = -0.0343$, $\sigma_{332} = -0.1928$
	$\ell_{31} = 0.3237$, $\ell_{32} = -0.0103$, $\ell_{33} = 0.2126$
	$\Gamma_{31} = 0.1101$, $\Gamma_{32} = -0.2328, \varphi_3 = -0.2245$
4	$\sigma_{411} = 0.2218$, $\sigma_{412} = 0.1624$
	$\sigma_{421} = 0.1328$, $\sigma_{422} = 0.1692$
	$\sigma_{431} = -0.0631$, $\sigma_{432} = 0.0634$
	$\ell_{41} = 0.2512$, $\ell_{42} = -0.1850$, $\ell_{43} = 0.2543$
	$\Gamma_{41} = 0.1313$, $\Gamma_{42} = -0.0582$, $\varphi_4 = 0.0580$

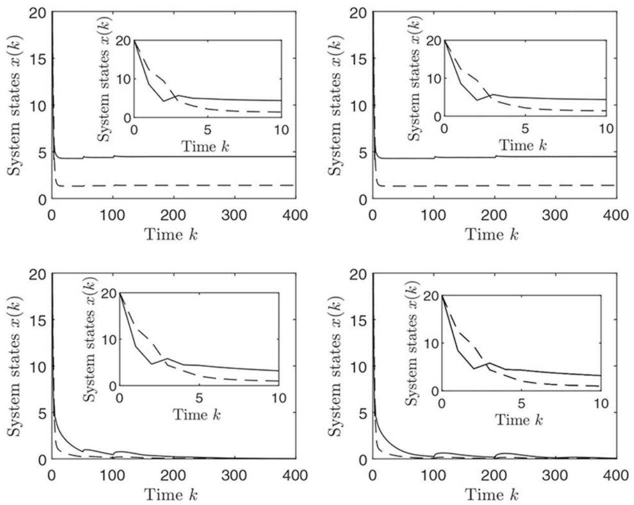

Fig. 3 *Top left and right figures are transient response of open-loop system states with time delay* $\tau_1 = 50, \tau_2 = 100$ *and* $\tau_1 = 100, \tau_2 = 200$*, respectively. The bottom left and right figures are transient response of closed-loop system states (feasible regions indicated by the symbols '×' based on Theorem 2 referring to Fig. 2), with time delay* $\tau_1 = 50, \tau_2 = 100$ *and* $\tau_1 = 100, \tau_2 = 200$*, respectively, and parameters all chosen for* $a = 6.5$*;* $b = 4.9$

Table 2 Upper and lower boundary of membership functions

Symbol	Parameters referring from (26)–(31)
$\underline{\eta}_i$	$\underline{\eta}_1 = 0.0000$, $\underline{\eta}_2 = 0.0025$, $\underline{\eta}_3 = 0.0000$
$\overline{\eta}_i$	$\overline{\eta}_1 = 0.9975$, $\overline{\eta}_2 = 0.9640$, $\overline{\eta}_3 = 0.9975$
$\underline{\varphi}_j$	$\underline{\varphi}_1 = 0.0002$, $\underline{\varphi}_2 = 0.0000$
$\overline{\varphi}_j$	$\overline{\varphi}_1 = 1.0000$, $\overline{\varphi}_2 = 0.9998$
$\underline{\rho}_{ij}$	$\underline{\rho}_{11} = 0.0000$, $\underline{\rho}_{12} = 0.0000$, $\underline{\rho}_{21} = 0.0000$
	$\underline{\rho}_{22} = 0.0000$, $\underline{\rho}_{31} = 0.0000$, $\underline{\rho}_{32} = 0.0000$
$\overline{\rho}_{ij}$	$\overline{\rho}_{11} = 0.1363$, $\overline{\rho}_{12} = 0.9973$, $\overline{\rho}_{21} = 0.9640$
	$\overline{\rho}_{22} = 0.3989$, $\overline{\rho}_{31} = 0.1363$, $\overline{\rho}_{32} = 0.9973$

into the stability analysis leads to more relaxation of the stability conditions and increasing the probability of finding a feasible solution.

To verify the feasible regions which can guarantee the stability and positivity of discrete time PFMB control system with time delay, in terms of feasible regions under different setting value of R shown in Fig. 2, the transient response of system states $\mathbf{x}(k)$ with initial conditions $\phi(0) = [20, 20]^T$ is conducted. Firstly, the open-loop transient response of system states $x_1(k)$ and $x_2(k)$ for $a = 6.5$, $b = 4.9$ with time delay $\tau_1 = 50$, $\tau_2 = 100$ and $\tau_1 = 100$, $\tau_2 = 200$, respectively, are obtained (cf. top left and right sub-figures in Fig. 3). As seen from top left and right sub-figures in Fig. 3, the original open-loop discrete-time system is unstable. Referring to the obtained feasible regions based on Theorem 2, $a = 6.5$, $b = 4.9$ indicated by '×' in Fig. 2 can be employed to guarantee the stability and positivity of discrete time PFMB control system with time delay. In order to verify the result, the closed-loop transient response of system states $x_1(k)$ and $x_2(k)$ for $a = 6.5$, $b = 4.9$ with time delay $\tau_1 = 50$, $\tau_2 = 100$ and $\tau_1 = 100$, $\tau_2 = 200$, respectively, are obtained (cf. bottom left and right sub-figures in Fig. 3). The corresponding PFMB controllers are listed in Table 3. The transient response of closed-loop system states is shown that the PFMB controller can stabilise the system and guarantee the positivity.

The phase plots of $x_1(k)$ and $x_2(k)$ are simulated with eight different initial conditions indicated by '∘' including $\phi(0) = [0, 9]^T, [0, 18]^T, [10, 5]^T, [6, 20]^T, [20, 15]^T, [8, 10]^T, [18, 6]^T, [20, 20]^T$. These results (see Fig. 4) show the original open-loop discrete time system with $a = 6.5$, $b = 4.9$ is unstable. The PFMB controller which is obtained from the feasible regions indicated by '×' with $a = 6.5$, $b = 4.9$ in Fig. 2 is able to drive all the system states to equilibrium (origin indicated by '◊' in Fig. 4) while always hold them positive based on different initial conditions.

In Figs. 5 and 6, we follow the same procedure with the same initial conditions to obtain the transient response and phase plots of system states $\mathbf{x}(k)$ for the open-loop discrete-time model with time delay $\tau_1 = 50$, $\tau_2 = 100$ and $\tau_1 = 100$, $\tau_2 = 200$, respectively (cf. top left and right sub-figures in Figs. 5 and 6 when parameters chosen as $a = 8.5$, $b = 6.3$. The feasible regions indicated by '□' with $a = 8.5$, $b = 6.3$ in Fig. 2 are chosen to obtain the transient response of system states and phase plots as well. The transient response and phase plots of system states for the closed-loop discrete time PFMB controller with time delay $\tau_1 = 50$, $\tau_2 = 100$ and $\tau_1 = 100$, $\tau_2 = 200$, respectively, are obtained (cf. bottom left

Table 3 Polynomial fuzzy controller based on Theorem 2 referring to Fig. 2

$\mathbf{G}_j(\mathbf{x}(k))$	Parameters for polynomial fuzzy controller with $a = 6.5$; $b = 4.9$ and $R = 2$
$\mathbf{G}_1(\mathbf{x}(k))$	$0.2817\times10^{-6}x_1^4 - 0.5131 \times 10^{-5}x_1^3 + 0.3450\times10^{-3}x_1^2 - 0.4895\times10^{-3}x_1 - 0.1078,$ $0.1017\times10^{-6}x_1^4 + 0.9506\times10^{-6}x_1^3 - 0.2073\times10^{-3}x_1^2 + 0.1537\times10^{-2}x_1 + 0.8714\times10^{-1},$
$\mathbf{G}_2(\mathbf{x}(k))$	$0.2832\times10^{-6}x_1^4 - 0.5658 \times 10^{-5}x_1^3 + 0.2782\times10^{-3}x_1^2 + 0.1690\times10^{-3}x_1 - 0.1372,$ $0.1017\times10^{-6}x_1^4 + 0.1653\times10^{-5}x_1^3 - 0.2397\times10^{-3}x_1^2 + 0.1129\times10^{-2}x_1 + 0.5773\times10^{-1}.$
$\mathbf{G}_j(\mathbf{x}(k))$	parameters for polynomial fuzzy controller with $a = 8.5$; $b = 6.3$ and $R = 3$
$\mathbf{G}_1(\mathbf{x}(k))$	$0.4741\times10^{-8}x_1^4 - 0.1652 \times 10^{-5}x_1^3 + 0.8661\times10^{-3}x_1^2 - 0.1073\times10^{-2}x_1 - 0.2032,$ $-0.7066\times10^{-9}x_1^4 - 0.2661\times10^{-6}x_1^3 - 0.1036\times10^{-2}x_1^2 + 0.2483\times10^{-2}x_1 + 0.2074\times10^{-1},$
$\mathbf{G}_2(\mathbf{x}(k))$	$0.6686\times10^{-8}x_1^4 - 0.1444 \times 10^{-5}x_1^3 + 0.8287\times10^{-3}x_1^2 - 0.3654\times10^{-2}x_1 - 0.1885,$ $-0.7274\times10^{-9}x_1^4 - 0.7636\times10^{-7}x_1^3 - 0.2704\times10^{-4}x_1^2 + 0.4778\times10^{-3}x_1 + 0.2448\times10^{-1}.$
$\mathbf{G}_j(\mathbf{x}(k))$	parameters for polynomial fuzzy controller with $a = 8.5; b = 6.7$ and R = 4
$\mathbf{G}_1(\mathbf{x}(k))$	$0.6449\times10^{-8}x_1^4 - 0.1512 \times 10^{-5}x_1^3 + 0.8135\times10^{-3}x_1^2 - 0.8008\times10^{-3}x_1 - 0.2078,$ $-0.2614\times10^{-9}x_1^4 - 0.2597\times10^{-6}x_1^3 - 0.2527\times10^{-4}x_1^2 + 0.2368\times10^{-2}x_1 - 0.5647\times10^{-2},$
$\mathbf{G}_2(\mathbf{x}(k))$	$0.5441\times10^{-8}x_1^4 - 0.1409 \times 10^{-5}x_1^3 + 0.8113\times10^{-3}x_1^2 - 0.1643\times10^{-2}x_1 - 0.1984,$ $0.2585\times10^{-11}x_1^4 - 0.3007\times10^{-6}x_1^3 - 0.3328\times10^{-4}x_1^2 + 0.2160\times10^{-2}x_1 + 0.5178\times10^{-2}.$

and right sub-figures in Figs. 5 and 6). The corresponding PFMB controllers are also listed in Table 3.

Then we choose the parameters as $a = 8.5$, $b = 6.7$, to plot the transient response and phase plots of system states $\mathbf{x}(k)$ for the open-loop discrete-time model with time delay $\tau_1 = 50$, $\tau_2 = 100$ and $\tau_1 = 100$, $\tau_2 = 200$, respectively (cf. top left and right sub-figures in Figs. 7 and 8), and choose the feasible regions indicated by '∘' with $a = 8.5$, $b = 6.7$ in Fig. 2 to obtain the transient response of system states and phase plots. The transient response and phase plots of the closed-loop discrete-time PFMB control system with time delay $\tau_1 = 50$, $\tau_2 = 100$ and $\tau_1 = 100$, $\tau_2 = 200$, respectively, are also obtained (cf. bottom left and right sub-figures in Figs. 7 and 8).

Comparing the open-loop and closed-loop results in Figs. 5 to 8 show that the closed-loop PFMB controller designed from the feasible regions indicated by '□' and '∘' can also guarantee the positivity and drive all the system states to equilibrium (origin indicated by '◇' in Figs. 6 and 8) from any initial conditions while the original open-loop discrete-time model cannot provide the asymptotic stability. Interestingly when we change the values of time delays $\tau_1 = 50$ to $\tau_1 = 100$ (cf. bottom left sub-figure in Figs. 4, 6 and 8), and $\tau_2 = 100$ to $\tau_2 = 200$ (cf. bottom right sub-figure in Figs. 4, 6 and 8), the system become stable regardless of the values of time delays. This is mainly because the stability conditions in Theorem 2 are independent of delay period.

5 Conclusions

This paper investigates the positivity and stability of discrete-time PFMB control system with time delay. The controller is designed based on IPC design concept to allow the number of fuzzy rules and membership function shape of the fuzzy model to be chosen

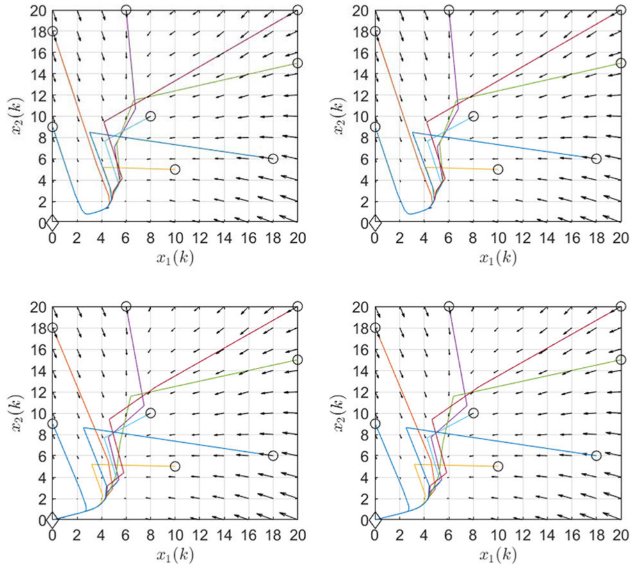

Fig. 4 *Top left and right phase plots are open-loop system states with time delay $\tau_1 = 50, \tau_2 = 100$ and $\tau_1 = 100, \tau_2 = 200$, respectively. The bottom left and right phase plots are closed-loop system states (feasible regions indicated by the symbols '×' based on Theorem 2 referring to Fig. 2), with time delay $\tau_1 = 50$, $\tau_2 = 100$ and $\tau_1 = 100$, $\tau_2 = 200$, respectively, and parameters all chosen for $a = 6.5$; $b = 4.9$*

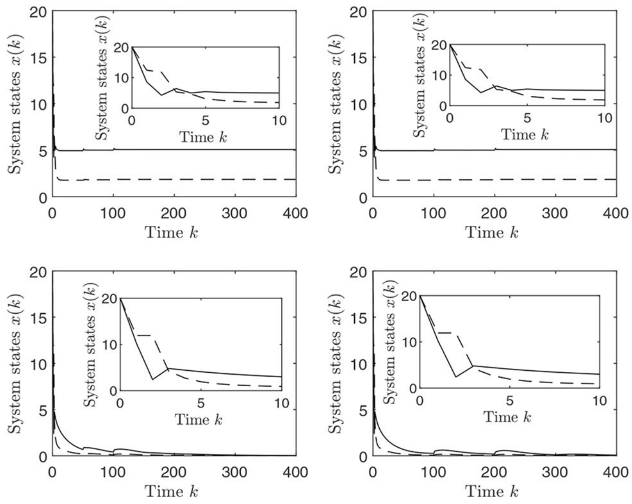

Fig. 5 *Top left and right figures are transient response of open-loop system states with time delay $\tau_1 = 50, \tau_2 = 100$ and $\tau_1 = 100, \tau_2 = 200$, respectively. The bottom left and right figures are transient response of closed-loop system states (feasible regions indicated by the symbols '□' based on Theorem 2 referring to Fig. 2), with time delay $\tau_1 = 50, \tau_2 = 100$ and $\tau_1 = 100, \tau_2 = 200$, respectively, and parameters all chosen for $a = 8.5; b = 6.3$*

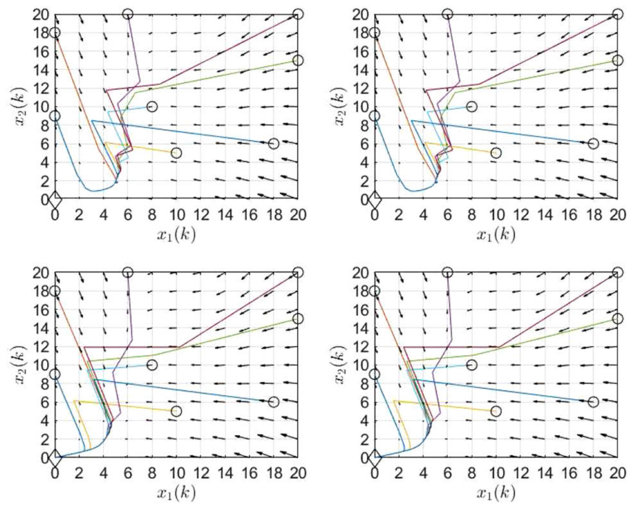

Fig. 6 *Top left and right phase plots are open-loop system states with time delay $\tau_1 = 50, \tau_2 = 100$ and $\tau_1 = 100, \tau_2 = 200$, respectively. The bottom left and right phase plots are closed-loop system states (feasible regions indicated by the symbols '□' based on Theorem 2 referring to Fig. 2), with time delay $\tau_1 = 50, \tau_2 = 100$ and $\tau_1 = 100, \tau_2 = 200$, respectively, and parameters all chosen for $a = 8.5; b = 6.3$*

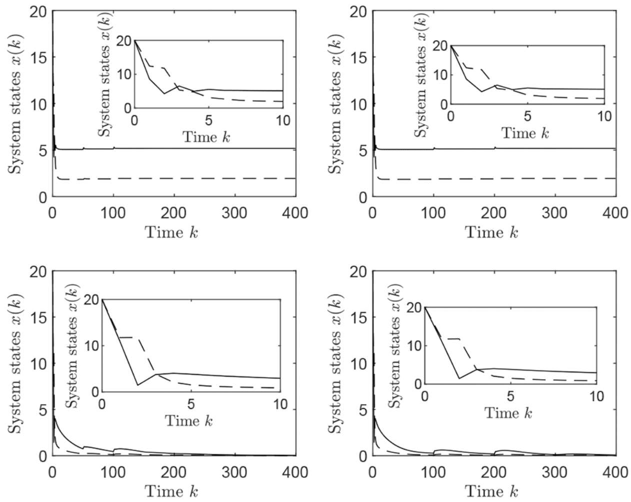

Fig. 7 *Top left and right phase plots are open-loop system states with time delay* $\tau_1 = 50, \tau_2 = 100$ *and* $\tau_1 = 100, \tau_2 = 200$*, respectively. The bottom left and right phase plots are closed-loop system states (feasible regions indicated by the symbols '∘' based on Theorem 2 referring to Fig. 2), with time delay* $\tau_1 = 50, \tau_2 = 100$ *and* $\tau_1 = 100, \tau_2 = 200$*, respectively, and parameters all chosen for* $a = 8.5; b = 6.7$

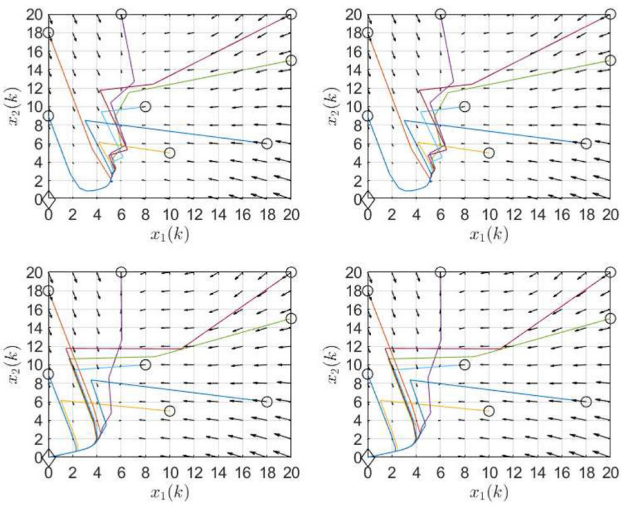

Fig. 8 *Top left and right phase plots are open-loop system states with time delay* $\tau_1 = 50$*,* $\tau_2 = 100$ *and* $\tau_1 = 100$*,* $\tau_2 = 200$*, respectively. The bottom left and right phase plots are closed-loop system states (feasible regions indicated by the symbols '∘' based on Theorem 2 referring to Fig. 2), with time delay* $\tau_1 = 50$*,* $\tau_2 = 100$ *and* $\tau_1 = 100$*,* $\tau_2 = 200$*, respectively, and parameters all chosen for* $a = 8.5$*;* $b = 6.7$

independently from those of the controller. This allows the information of membership functions, i.e. (i) the boundary information of membership functions of the model and the controller, and (ii) the relationship constraint information of the membership functions between the model and controller, to be introduced in the stability analysis in form of slack matrices. As a result, we relax the conservative formulation for the asymptotic stability and positivity conditions described in Theorems 1 and 2. We have shown introducing more numbers of the relationship constraint of the membership functions between model and controller into the stability analysis will further relax the stability and positivity formulations.

In terms of future research direction, the stability and positivity conditions of discrete time PFMB control system with time delay can be further relaxed by considering the information premise variables in the proposed stability and positivity theorem. The time delay constant term in this paper can be extended into time-varying delay terms. Furthermore, the proposed fuzzy co-positive Lyapunov function stability analysis can be employed to control discrete time positive non-linear system with time delay by combining other control methods such as output-feedback and observer-based feedback controller. Furthermore, both widely application in communication systems and formation flying and theoretical challenge of switched positive systems with time delays show a big motivation to study such different kinds of system.

6 Acknowledgments

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7 References

[1] Tian, Y., Sun, Y.: 'Stability analysis of switched positive nonlinear systems by mode-dependent average dwell time method', *IEEE Access*, 2019, **7**, pp. 15584–15589

[2] Feng, S., Wang, J., Zhao, J.: 'Stability and robust stability of switched positive linear systems with all modes unstable', *IEEE/CAA J. Autom. Sin.*, 2019, **6**, (1), pp. 167–176

[3] Wu, H., Su, H.: 'Observer-based consensus for positive multiagent systems with directed topology and nonlinear control input', *IEEE Trans. Control Syst. Man Cybern. Syst.*, 2019, **49**, (7), pp. 1459–1469

[4] Oghbaee, A., Shafai, B., Nazari, S.: 'Complete characterisation of disturbance estimation and fault detection for positive systems', *IET Control Theory Appl.*, 2018, **12**, (7), pp. 883–891

[5] Shu, Z., Lam, J., Gao, H., *et al.*: 'positive observers and dynamic output-feedback controllers for interval Positive linear systems', *IEEE Trans. Circuits Syst. I, Regul. Pap.*, 2008, **55**, (10), pp. 3209–3222

[6] Liu, X.: 'Stability analysis of switched positive systems: a switched linear copositive Lyapunov function method', *IEEE Trans. Circuits Syst. II, Express Briefs*, 2009, **56**, (5), pp. 414–418

[7] Li, S., Xiang, Z., Guo, J.: 'Stabilisation for positive switched T−S fuzzy delayed systems under standard L_1 and L_∞ performance', *Int. J. Syst. Sci.*, 2018, **49**, (6), pp. 1226–1241

[8] Li, S., Xiang, Z.: 'Exponential stability analysis and L_2 −gain control synthesis for positive switched T–S fuzzy systems', *Nonlinear Anal. Hybrid Syst.*, 2018, **27**, pp. 77–91

[9] Lian, J., Li, S., Liu, J.: 'T–S fuzzy control of positive Markov jump nonlinear systems', *IEEE Trans. Fuzzy Syst.*, 2018, **26**, (4), pp. 2374–2383
[10] Zhang, C., He, Y., Jiang, L.*, et al.*: 'Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality', *IEEE Trans. Cybern.*, 2017, **47**, (10), pp. 3040–3049
[11] Cui, Y., Shen, J., Feng, Z.*, et al.*: 'Stability analysis for positive singular systems with time-varying delays', *IEEE Trans. Automat. Control*, 2018, **63**, (5), pp. 1487–1494
[12] Chen, J., Meng, S., Sun, J.: 'Stability analysis of networked control systems with aperiodic sampling and time-varying delay', *IEEE Trans. Cybern.*, 2017, **47**, (8), pp. 2312–2320
[13] Zhang, X., Han, Q.: 'delay-dependent robust H_∞ filtering for uncertain discrete-time systems with time-varying Delay based on a finite sum inequality', *IEEE Trans. Circuits Syst. II, Express Briefs*, 2006, **53**, (12), pp. 1466–1470
[14] Teng, L., Wang, Y., Cai, W.*, et al.*: 'Fuzzy model predictive control of discrete-time systems with time-varying delay and disturbances', *IEEE Trans. Fuzzy Syst.*, 2018, **26**, (3), pp. 1192–1206
[15] Zhang, Y., Xu, S., Zhang, B.: 'Robust output feedback stabilization for uncertain discrete-time fuzzy Markovian jump systems with time-varying delays', *IEEE Trans. Fuzzy Syst.*, 2009, **17**, (2), pp. 411–420
[16] Xie, X., Yue, D., Peng, C.: 'Relaxed real-time scheduling stabilization of discrete-time Takagi–Sugeno fuzzy systems via an alterable-weights-Based ranking switching mechanism', *IEEE Trans. Fuzzy Syst.*, 2018, **26**, (6), pp. 3808–3819
[17] Xie, X., Yue, D., Peng, C.: 'Observer design of discrete-time fuzzy systems based on an alterable weights method', *IEEE Trans. Cybern.*, 2020, **50**, (4), pp. 1430–1439
[18] Zhang, Y., Tao, G., Chen, M.: 'Relative degrees and adaptive feedback linearization control of T–S fuzzy systems', *IEEE Trans. Fuzzy Syst.*, 2015, **23**, (6), pp. 2215–2230
[19] Chen, B., Liu, X., Tong, S.*, et al.*: 'Observer-based stabilization of T–S fuzzy systems with input delay', *IEEE Trans. Fuzzy Syst.*, 2008, **16**, (3), pp. 652–663
[20] Lam, H.K., Tsai, S.H.: 'Stability analysis of polynomial-fuzzy-model-based control systems with mismatched premise membership functions', *IEEE Trans. Fuzzy Syst.*, 2014, **22**, (1), pp. 223–229
[21] Lam, H.K., Liu, C., Wu, L.*, et al.*: 'Polynomial fuzzy-model-based control systems: stability analysis via approximated membership functions considering sector nonlinearity of control input', *IEEE Trans. Fuzzy Syst.*, 2015, **23**, (6), pp. 2202–2214
[22] Song, G., Lam, H., Yang, X.: 'Membership-function-dependent stability analysis of interval type-2 polynomial fuzzy-model-base control systems', *IET Control Theory Appl.*, 2017, **11**, (17), pp. 3156–3170
[23] Prajna, S., Papachristodoulou, A., Parrilo, P.A.: 'Introducing SOSTOOLS: a general purpose sum of squares programming solver'. Proc. of 41st IEEE Conf. Decision and Control 2002, Las Vegas, NV, USA, 2002, vol. 1, pp. 741–746
[24] Lam, H.K.: 'A review on stability analysis of continuous-time fuzzy-model-based control systems: from membership-function-independent to membership-function-dependent analysis', *Eng. Appl. Artif. Intell.*, 2018, **67**, pp. 390–408
[25] Lam, H.K.: '*Polynomial fuzzy model-based control systems*' (Springer International, Switzerland, 2016)
[26] Tanaka, K., Ikeda, T., Wang, H.O.: 'Fuzzy regulators and fuzzy observers: relaxed stability conditions and lmi-based designs', *IEEE Trans. Fuzzy Syst.*, 1998, **6**, (2), pp. 250–265
[27] Mao, Y., Zhang, H., Qin, Y.*, et al.*: 'Stability and constrained control of a class of discrete-time fuzzy positive systems with time-varying delays', *Circuits Syst. Signal Process*, 2013, **32**, (2), pp. 889–904
[28] Benzaouia, A., Oubah, R., El-Hajjaji, A.: 'Stabilization of positive takagi-sugeno fuzzy discrete-time systems with multiple delays and bounded controls', *J. Franklin Inst. Eng. Appl. Math.*, 2014, **351**, (7), pp. 3719–3733
[29] Narimani, M., Lam, H.K.: 'SOS-based stability analysis of polynomial fuzzy-model-based control systems via polynomial membership functions', *IEEE Trans. Fuzzy Syst.*, 2010, **18**, (5), pp. 862–871
[30] Xiao, B., Lam, H.K., Li, H.: 'Stabilization of interval type-2 polynomial-fuzzy-model-based control systems', *IEEE Trans. Fuzzy Syst.*, 2017, **25**, (1), pp. 205–217
[31] Lam, H.K., Seneviratne, L.D.: 'Stability analysis of polynomial fuzzy-model-based control systems under perfect/imperfect premise matching', *Control Theory Appl. IET*, 2011, **5**, (15), pp. 1689–1697
[32] Lam, H.K., Narimani, M.: 'Stability analysis and performance design for fuzzy-model-based control system under imperfect premise matching', *IEEE Trans. Fuzzy Syst.*, 2008, **17**, (4), pp. 949–961
[33] Lam, H.K., Narimani, M.: 'Quadratic-stability analysis of fuzzy-model-based control systems using staircase membership functions', *IEEE Trans. Fuzzy Syst.*, 2009, **18**, (1), pp. 125–137
[34] Lam, H.K.: 'Polynomial fuzzy-model-based control systems: stability analysis via piecewise-linear membership functions', *IEEE Trans. Fuzzy Syst.*, 2011, **19**, (3), pp. 588–593
[35] Benzaouia, A., Hajjaji, A.E., Hmamed, A.: 'Relaxed stabilization of controlled positive discrete-time T-S fuzzy systems'. Proc. of 2010 18th Mediterranean Conf. Control Automation (MED), Marrakech, Morocco, 2010, pp. 1407–1412
[36] Benzaouia, A., Oubah, R.: 'Stability and stabilization by output feedback control of positive takagi-Sugeno fuzzy discrete-time systems with multiple delays', *Nonlinear Anal. Hybrid Syst.*, 2014, **11**, pp. 154–170
[37] Li, X., Merhan, K.: 'Model-based control and stability analysis of discrete-time polynomial fuzzy systems with time delay and positivity constraints', *IEEE Trans. Fuzzy Syst.*, 2019, **27**, (11), pp. 2090–2100
[38] Li, X., Lam, H.K., Liu, F.*, et al.*: 'Stability and stabilization analysis of positive polynomial fuzzy systems with time delay considering piecewise membership functions', *IEEE Trans. Fuzzy Syst.*, 2017, **25**, (4), pp. 958–971
[39] Li, X., Lam, H.K., Song, G.*, et al.*: 'Stability analysis of positive polynomial fuzzy-model-based control systems with time delay under imperfect premise matching', *IEEE Trans. Fuzzy Syst.*, 2018, **26**, (4), pp. 2289–2300
[40] Li, X., Liu, C., Lam, H.K.*, et al.*: 'Stability analysis of fuzzy polynomial positive systems with time delay'. 2014 Int. Conf. on Fuzzy Theory and Its Applications (iFUZZY2014), Kaohsiung, Taiwan, 2014, pp. 24–28