Kansei Engineering and Soft Computing:
Theory and Practice

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Chapter 8
Fuzzy Logic for Non-Smooth Dynamical Systems

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ABSTRACT

Dynamical system theory has proved to be a powerful tool in the analysis and comprehension of a diverse range of problems. Over the past decade, a significant proportion of these systems have been found to contain terms that are non-smooth functions of their arguments. These problems arise in a number of practical systems ranging from electrical circuits to biological systems and even financial markets. It has also been demonstrated that Fuzzy engineering can be effectively employed to identify or even predict an array of uncertainties and chaotic phenomena caused by discontinuities typical of this class of system. This chapter presents a review of the most recent developments concerned with the confluence of these two fields through real-life examples and current advances in research.

1. INTRODUCTION

Bridging the two seemingly unrelated concepts, fuzzy logic and nonlinear piecewise-smooth dynamical systems theory is chiefly motivated by the concept of soft computing (SC), initiated by Lotfi A. Zadeh, the founder of fuzzy set theory. The principal components of SC, as defined in his initiative for soft computing[1], are fuzzy logic (FL), neural network theory (NN) and probabilistic reasoning (PR), with the latter subsuming parts of belief networks, genetic algorithms, chaos theory and learning theory. SC is essentially distant from traditional, immutable (hard) computing and is much more aligned to the main ideas of Kansei Engineering in that the imprecision, uncertainty and partial truth reflecting the working of the human mind are incorporated into the computing process to form a new paradigm to tackle highly complex, nonlinear systems. The aim of this chapter is to combine FL from general Soft Computing theory and piece-wise smooth dynamical systems from

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the general theory of nonlinear dynamical systems. The main target being to examine their relationship and interaction and to demonstrate that the blending of the two concepts can be effectively used to analyse and control the chaotic and other nonlinear behaviours typical of such systems.

Over the past few decades, there has been a substantial level of interest in fuzzy systems technology and dynamical systems theory shown by almost all hard and soft-science research communities such as theoretical and experimental physicists, applied mathematicians, meteorologists, climatologists, physiologists, psychologists and engineers. More specifically, fuzzy system technology has emerged as an effective methodology to solve many problems ranging from control engineering, robotics, and automation to system identification, medical image/signal processing and Kansei engineering. Meanwhile, dynamical systems theory has proved to be a powerful tool to analyze and understand the behaviour of a diverse range of real life problems. The vast majority of these problems can only be modelled with dynamical systems whose behaviour is characterized by instantaneous changes and discontinuities. These practically ubiquitous dynamical systems are usually referred to as *piece-wise smooth* or *non-smooth* dynamical systems. Examples include mechanical systems with friction, robotic systems, electric and electronic systems employing electronic switching devices, unmanned vehicle systems, biological and biomedical systems, climate modelling and even financial forecasting. The study of these systems is of a great importance, primarily because they have captivating dynamics with significant practical applications and show rich nonlinear phenomena such as *quasi-periodicity* and *chaos*.

Unfortunately, many of the mathematical tools developed for smooth dynamical systems have to-date proven inadequate when dealing with the discontinuities present in non-smooth systems. Furthermore, certain specific approaches are not well-established and are still in their early stages of development. Not surprisingly, fuzzy system technology, in the context of highly complex nonlinear systems, can be critically intuitive and useful. This belief arises from the fact that fuzzy logic resembles human reasoning in its use of approximate information so it can embody the uncertainty which is the essential part of the mainly event-driven, non-smooth system and its chaotic dynamics. That's why it is possible to believe that FL can lead to a general theory of uncertainty[2].

The chapter does not intend – in fact, is not able – to provide a thorough explanation of the intrinsic relationship between fuzzy logic and dynamical system theory, but attempts to give some heuristic research results and insightful ideas, shedding some light on the subject and attracting more attention to the topic.

The chapter is organised in the following way: it commences with an outline of the fundamental concepts of nonlinear dynamical systems theory, and specifically non-smooth dynamical systems, with examples showing the richness and uniqueness of their nonlinear behaviours. The inherent difficulties in modelling these dynamical systems will be highlighted. This is followed by an examination of how the Takagi-Sugeno fuzzy modelling concept could be extended to overcome these problems. The fuzzy approach is further extended and applied to the stability analysis (in the Lyapunov framework) to predict the onset of structural instability or so-called bifurcations in the evolution of the dynamical system. The chapter ends with an evaluation of the proposed approach in nonlinear system modelling and analysis in general and further discussion about potential forthcoming research.

Although this chapter may raise more questions than it can afford answers, we hope that it nevertheless will open more research avenues and plant the seeds for future developments in this area.

2. BACKGROUNDS

Non-Smooth Dynamical Systems

"Strictly speaking, there is no such thing as a non-smooth dynamical system" says Mario Di Bernardo[3], a leading expert on non-smooth dynamical systems. This might seem somewhat confusing at first, but the statement is true when considering the critical time-scales over which transitions occur between different systems topologies. For instance, in some impacting mechanical systems; the impact takes place over a very short time compared with that of the overall dynamics. Discontinuities or sudden jumps also happen in economic systems such as foreign exchange-rate markets (FOREX) and can lead to full-scale unstable chaotic behaviour, which cannot be understood if we model the dynamics of the market as a smooth system. In truly smooth systems such a scenario normally occurs after a long sequence of bifurcations, such as the Feigenbaum cascade of period-doubling bifurcations.

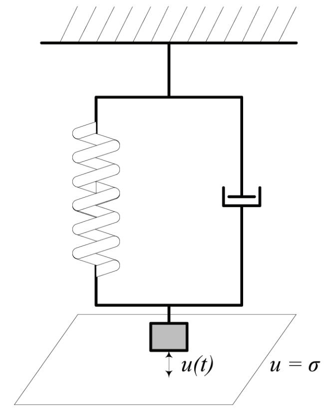
Another plausible reason for the failure of researchers to appreciate the necessity of establishing appropriate tools and techniques for piece-wise smooth dynamical systems, is that they expostulate many of our assumptions about nonlinear dynamics. Concepts like structural stability, bifurcation and qualitative measures of chaos are either indefinable or need to be redefined to explore the variety of nonlinear phenomena unique to non-smooth systems.

For these particular reasons, we will first outline some elementary but essential concepts of non-smooth dynamical systems and give some case studies and examples arising from different disciplines.

Case Study I: Impact Oscillator

An elastic ball bouncing on a table top is a very simple example of what mechanical engineers call an *impact oscillator,* i.e. a low-degree of freedom mechanical system with hard constraints that feature impacts (Babitskii, 1978; Feigin). Systems with impacting characteristics are ubiquitous in mechanical systems ranging from gear assemblies and car suspensions to walking robots and many body-particle dynamics (Di Bernardo, Budd, Kowalczyk, & Champneys, 2007). The effect of the impact makes these systems highly nonlinear to the degree that chaotic phenomena turns out to be an inevitable attribute. Figure 1 shows a simple sketch of an impact oscillator. If we think of the motion of a body in one dimension, described by the position $u(t)$ and velocity $v(t)=du/dt$ of its centre of mass, we can visualise this body as a single particle in space. Figure 1 shows that there is a linear spring and dashpot attaching this particle to a datum point in order that its position fulfils the dimensionless differential equation in free motion:

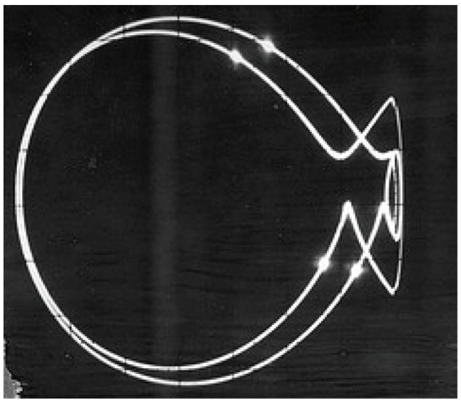
Figure 1. A simple impact oscillator



$$\frac{d^2u}{dt^2} + 2\zeta\frac{du}{dt} + u = w(t), \quad if \quad u > \sigma, \tag{2.1}$$

where, the mass and stiffness have been scaled to unity, 2ζ denotes the viscous damping coefficient, and $w(t)$ is an applied external force. We assume that motion is free in the region $u > \sigma$, until there is impact with the rigid surface $u = \sigma$ at some time

Figure 2. Oscilloscope trace of the phase portrait of the impact oscillator modelled by an electronic circuit, with the velocity variable v(t) shown vertically and the position variable u(t) horizontally. This experimental result agrees with the solution of (2.1)-(2.3) with σ = 0, r = 0.95, ζ = 0 and ω = 2.76 (M. Oestreich, Hinrichs, & Popp, 1996)



t_0. What actually happens, from the rigorous mathematical point of view, is that $(u(t_0), v(t_0)) := (u_-, v_-)$ is mapped at zero time to (u^+, v^+) via an *impact law*

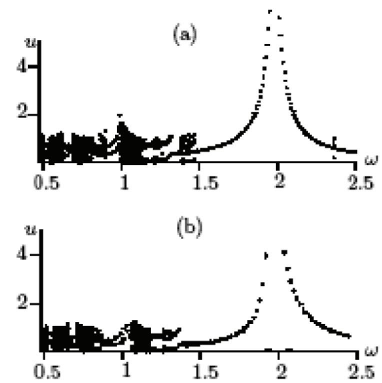
$$u^+ = u^- \quad and \quad v^+ = -rv^-, \qquad (2.2)$$

where $0 < r < 1$ is Newton's coefficient of restitution. The resulting phenomena in this system can be of course be studied for different types of external forcing function (Bishop, 1994). Here, for the sake of brevity, we focus on a typical sinusoidal forcing function given by:

$$w(t) = \cos(\omega t), \quad with\ period \quad T = \frac{2\pi}{\omega}, \qquad (2.3)$$

The occurrence of impacts introduces *discontinuities* and an ensuing fierce *nonlinearity* into the system; otherwise, the system can be modelled as a smooth system or even a linear system, for which there are well-established analysis tools. The nonlinearity makes the system acutely sensitive to its initial conditions to the extent that it shows highly irregular chaotic behaviour; not observed in any smooth nonlinear system (Leine & Nijmeijer, 2004). This sensitivity can be easily demonstrated by plotting the solution trajectories of (2.1)-(2.3) in the phase plane *(u,v)*, as shown in Figure 2.

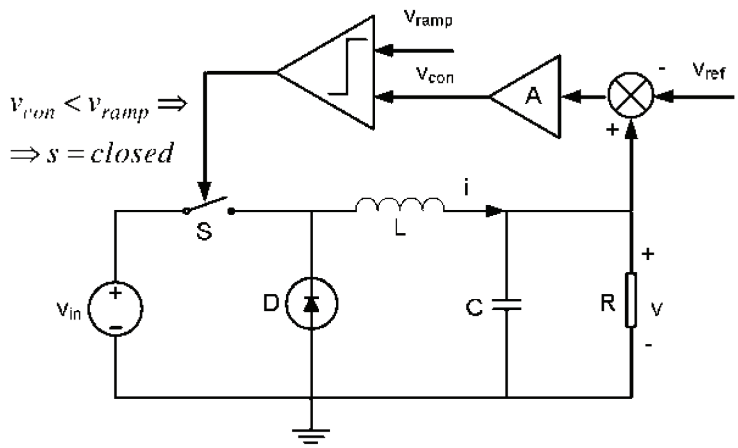
Figure 3. Bifurcation diagram for increasing ω; σ = 0 and r = 0.9 (a) Analytical and (b) experimental results (M Oestreich, Hinrichs, Popp, & Budd, 1997)



Evidently, the system shows complicated periodic motion intermittent with the chaotic motion at the intervals when we vary, say, the number of impacts per period ω, as a parameter. This can be better seen by the bifurcation diagram of Figure 3, where some measure of the solution state, say oscillator position, is plotted against the parameter ω. As one can notice, there is a striking agreement between the numerically computed bifurcation diagram, resulted from selecting a random set of initial data and solving (2.1)-(2.3) with a numerical software package like Matlab (M. Oestreich, et al., 1996), and the experimental traces resulting from stroboscopic sampling (M Oestreich, et al., 1997).

In a smooth nonlinear system, the characteristic behaviour is for a Fiegenbaum cascade of

Figure 4. Schematic diagram of a DC to DC buck converter



period-doubling bifurcation to (2^k, 2^k) orbits, for $k = 1,2, ..., \infty$ leading to chaos (Cvitanovic, 1984). This cannot be observed in the impact oscillator, mainly because the resulting orbit of the system loses stability through grazing bifurcations (Di Bernardo, et al., 2007), which can only be observed in non-smooth dynamical systems. That is why in such systems, an unexpectedly different level of complexity is introduced into the *attractors* of the dynamical system or the so-called *ω-limit* sets. In a grazing event, which in a physical sense is the time just before the impact with the rigid surface $u = \sigma$, a periodic attractor can be instantly transformed to a chaotic attractor. Although this chaotic attractor may arise in intervals, it is utterly different in nature from the *period-doubling cascade* associated with the chaotic behaviour of smooth dynamical systems.

Case Study II: Electronic Converters

DC to DC converters have long been regarded as a good example of a nonlinear dynamical system. This might be for the simplicity of its circuit topology but also the variety of observed complex nonlinear phenomena. These circuits are almost omnipresent in our life. The laptop computer and most hand-held devices we use everyday will have one of these circuits to convert DC voltage to another DC voltage in a cheap, energy efficient manner. However, what's really important in our case, is the many possible forms of nonlinear dynamical behaviours the converter can exhibit when operating beyond a set-limit range of their input voltage, output current or other circuit parameters. An ideal representative voltage-mode DC/DC buck converter is shown in Figure 4. What actually makes the converter a non-smooth system, is the toggling between two circuit topologies or, in other words, two sets of differential equations:

$$\frac{di(t)}{dt} = \begin{cases} \dfrac{v_{in} - v(t)}{L}, & S \text{ is blocking} \\ -\dfrac{v(t)}{L}, & S \text{ is conducting} \end{cases} \tag{2.4}$$

$$\frac{dv(t)}{dt} = \frac{i(t) - \dfrac{v(t)}{R}}{C} \tag{2.5}$$

As apparent from the above equations, the electronic switch S introduces a discontinuity into the right-hand side of (2.4). The system naturally

Figure 5. DC-DC converter bifurcation diagram when the input voltage is varied as a bifurcation parameter. The sudden transition from a periodic to a chaotic attractor, visible as the black band, is quite distinguishable from the ω-limit set of the system

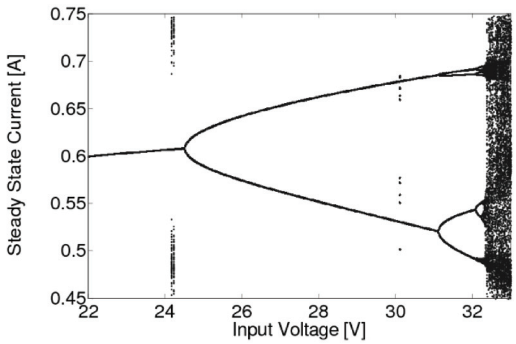
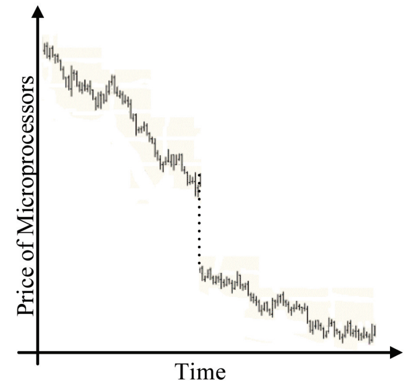


Figure 6. Discontinuity created in the price movement of microprocessors shows that it can be modelled as a non-smooth system



becomes a non-smooth dynamical system, referred to as a Fillipov system. Fillipov-type systems, which get their name from the 18th century Russian mathematician Alexander Fillipov who created the mathematical framework for studying discontinuous systems (Filippov, 1988), are non-smooth systems with a degree of smoothness equal to one (see section 3.1 for a definition).

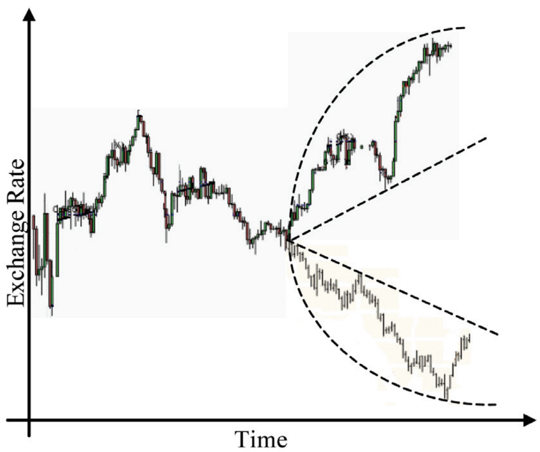
If, for instance, the input voltage goes beyond the nominal range of the converter, a *Discontinuity-induced bifurcations* (DIB) makes the system lose stability and jump into chaotic behaviour. DIBs are a bifurcation phenomena that occurs only in non-smooth dynamical systems (Di Bernardo, et al., 2007). This is clearly observable in Figure 5, in which a corner-collision bifurcation leads the system to a chaotic attractor sooner than expected in any counterpart smooth system.

Case Study III: Financial Markets

The financial market is in fact one of the most complex nonlinear dynamical system imaginable. Over the years, it's been a long-held assumption that complex phenomena are the outputs of the systems with many degrees of freedom and were analyzed as random process. Simple phenomena were always modelled deterministically. However, as we have learned from the previous case studies, only a few degrees of freedom are necessary to generate complex chaotic motion. One of the most critical points in analyzing and predicting market trends is how to deal with *discontinuities* and their impact.

From the nonlinear dynamical point of view, a discontinuity is an apparent jump caused by a sudden increase in the size of the main attractor (solution space). Any attempt to model such a system using methods derived from smooth function techniques would be doomed to failure. This is as true in economics as it is in electrical and mechanical systems. At the same time, nonlinear mathematics suggests that there are infinitely many possibilities for discontinuities or *non-smoothness,* i.e. many combinations of factors causing jumps. Though many of these jumps can be of small magnitude, they can end up causing a big change in the dynamics of the system. A simple example of this non-smooth behaviour or sudden jump can be seen in the price movement of semiconductor chips (see Figure 6). When a new chip is introduced to the market, the price is relatively high. For the next year or so, the price fluctuates, dropping slowly. However, when the

Figure 7. Attenuating and amplifying of nonlinear factor f in the logistic equation of time series $Z_{(t+1)} = f \cdot Z_t$, where f itself a variable influenced by bid and ask or $f = aX + bY$ can lead to a DIB and more complex behaviour



next generation of chip comes out, the price of the old chip would drop sharply before it starts fluctuating again.

This non-smooth behaviour is somehow ever-present in all sorts of markets. One good example is the behaviour of Foreign Exchange Markets or FOREX. This is a currency market where banks and other financial institutions facilitate currency trading, i.e. the buying and selling of foreign currencies. Forex is by far one the most volatile and liquid market in the world, with large numbers of market participations (O'Sullivan & Sheffrin, 2003). One of the evident qualitative factors in this market is *market psychology;* in other words, rumours, trends, envy and greed. There is also a feedback effect in a financial market setting. Markets may have no long-term memory, but individuals who might have burned their fingers or made a fortune form past events have lots of it. If all these factors are considered in a logistic equation, mapping or projecting the time series of the Forex market, the effects of a severe fluctuation (attenuation or amplification) and discontinuity-induced bifurcation (DIB) can be visible.

In October 1992, speculation on the dollar/yen exchange rate attenuated, when the attack on the French Franc verses Deutsche Mark parity took all of the Forex market attention, forcing a *discontinuity* in the parity trend, as it did in the same period with the British Pound and the Italian Lira, devaluated by 20 and 30 percent, respectively. A DIBs of the exchange rate price may result, as happened in the early 1980s with the Belgian Franc against other major currencies at the time, which had two values: commercial and financial, both fluctuating with market movements (Figure 7).

Fuzzy Engineering

Informed by the previous case studies, the challenge is how to put known procedures into effect when systems are dynamic and their output presents discontinuities (or non-smoothness) as well as uncertainties[4]. Uncertainties occur when we are not absolutely clear about the information elements at our disposal. The degree of certainty is usually represented by a *quantitative* value. In contrast, most problems encountered in dynamical systems are *qualitative*.

Fuzziness exists when the boundary of information is not clear-cut. Expression such as "more or less", "roughly equal to", "somewhat greater than average" are good examples of this. In some situations, *uncertainty* and *vagueness* may occur simultaneously, and precisely because of this, not every mathematical tool can serve the purpose of tackling piecewise dynamical systems – nor can even the most sophisticated analysis be applied in every situation. For example,

- A polynomial fit that oscillates widely between data points that appear to lie on a *smooth* curve would be a bad choice for *approximating* non-smooth dynamical functions.
- A tool that presupposes crisp (0 and 1 type) data or situations will not answer the chal-

lenges posed by vague conditions involving a significant measure of uncertainty.

The concept of Fuzzy logic, proposed by Zadeh in 1965, is essentially about how to deal with uncertainties. This is made clearer from his recent attempt (Zadeh, 2005) to expand the whole idea to form a generalized theory which can explain and quantify various different forms of uncertainties in existing systems. Uncertainty being the centrepiece of the theory is of prime importance, mainly because it includes both probability and possibility as special constraints.

Fuzzy engineering, in this respect, can really assist in pinpointing these seemingly unpredictable behaviours and identify their overall dynamics. The first stage in this approach is *system identification*. Among the different fuzzy system identification, the Takagi-Sugeno (TS) fuzzy modelling approach (Takagi & Sugeno, 1985) has been more successful in modelling the dynamics of complex nonlinear systems.

Let's understand better the whole modelling procedure of this approach by approximating a simple nonlinear system as an example. Consider the nonlinear system:

$$\begin{cases} \dot{x}_1 = x_2 + 3 \\ \dot{x}_2 = x_1^2 + x_2^2 + 3 + u \end{cases} \tag{2.6}$$

The ultimate goal of modelling the above nonlinear system through the TS fuzzy approach is that the response of the system as a black box with the input u would be the same as the original system. We specify a domain for variables $x_1 \in [0.5, 3.5]$ and $x_2 \in [-1, 4]$ as the area of fuzzy approximation and take advantage of exact linearization as the modelling method (the detailed general procedure of obtaining TS fuzzy model using this method can be found in (Tanaka & Wang, 2001)). The system (2.6) can be equivalently represented as:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ x_1 & x_2 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ 3 + u \end{bmatrix}$$

where $x(t)=[x_1(t)\, x_2(t)]^T$. As x_1 and x_2 are nonlinear terms, we choose our fuzzy variables (premise variables) as function of state variables:

$$\begin{cases} z_1 = x_1 \\ z_2 = x_2 \end{cases}$$

The domain of premise variables z_1 and z_2 can be easily obtained from the domain of x_1 and x_2. Therefore, these can be represented by a combination of membership functions:

$$\begin{cases} z_1 = M_1(z_1) \cdot 3.5 + M_2(z_1) \cdot 0.5 \\ z_2 = N_1(z_1) \cdot 4 + N_2(z_1) \cdot (-1) \end{cases},$$

where M_1, M_2, N_1 and N_2 are fuzzy sets, on which the following rules apply

$$\begin{cases} M_1(z_1) + M_2(z_1) = 1 \\ N_1(z_1) + N_2(z_1) = 1 \end{cases},$$

If we name the fuzzy sets respectively as "Positive", "Negative", "Big" and "Small", the rules of the fuzzy system or the so-called model rules can be expressed by a typical fuzzy IF-THEN rule base as:

Model Rule 1:
IF $z_1(t)$ is "Positive" AND $z_2(t)$ is "Big",
THEN $\dot{x}(t) = A^1 x(t) + B^1$
Model Rule 2:
IF $z_1(t)$ is "Positive" AND $z_2(t)$ is "Small",
THEN $\dot{x}(t) = A^2 x(t) + B^2$
Model Rule 3:
IF $z_1(t)$ is "Negative" AND $z_2(t)$ is "Big",
THEN $\dot{x}(t) = A^3 x(t) + B^3$

Model Rule 4:
IF $z_1(t)$ is "Positive" AND $z_2(t)$ is "Small",
THEN $\dot{x}(t) = A^4 x(t) + B^4$,

where the subsystems are determined as:

$$A^1 = \begin{bmatrix} 0 & 1 \\ 3.5 & 4 \end{bmatrix},\ A^2 = \begin{bmatrix} 0 & 1 \\ 3.5 & -1 \end{bmatrix},$$
$$A^3 = \begin{bmatrix} 0 & 1 \\ 0.5 & 4 \end{bmatrix},\ A^4 = \begin{bmatrix} 0 & 1 \\ 0.5 & -1 \end{bmatrix} \text{ and}$$
$$B^1 = B^2 = B^3 = B^4 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The truth value (or activation degree) h^j for the complete rule j is computed using the aggregation operator AND, also called a *t-norm*, often denoted by

$\otimes:[0,1]\times[0,1]\rightarrow[0,1]$,

$h^1(z)=M^1(z_1)\otimes N^1(z_2)$

$h^2(z)=M^1(z_1)\otimes N^2(z_2)$

$h^3(z)=M^2(z_1)\otimes N^1(z_2)$

$h^4(z)=M^2(z_1)\otimes N^2(z_2)$

Finally after the defuzzification process, the TS fuzzy model is derived as:

$$\dot{x}(t) = \sum_{j=1}^{4} h^j(z(t))(A^j x(t) + B^j u(t)) \qquad (2.7)$$

The model (2.7) can accurately represents the nonlinear system (2.6) in the region [0.5, 3.5]´[-1,4] in the x_1-x_2 space.

The best choice of the consequent functions (or fuzzy sub-systems) would be the *affine* functions as seen in (2.7); therefore the TS fuzzy model can be generally explained in the form:

$$\dot{x} = \sum_{j=1}^{l} w^j(\theta)(A^j x + B^j u + a^j) = A(\theta)x + B(\theta)u + a(\theta)$$
$$y = \sum_{j=1}^{l} w^j(\theta)(C^j x + c^j) = C(\theta)x + c(\theta), \qquad (2.8)$$

where $A^j:\mathbb{R}^{n\times n}$, $B^j:\mathbb{R}^{n\times m}$, $a^j\in\mathbb{R}^n$, $C^j:\mathbb{R}^{p\times n}$, $c^j\in\mathbb{R}^p$ and $w^j(\theta)$ is the normalized degree of activation h^j for each model rule j, which we call the weighting functions.

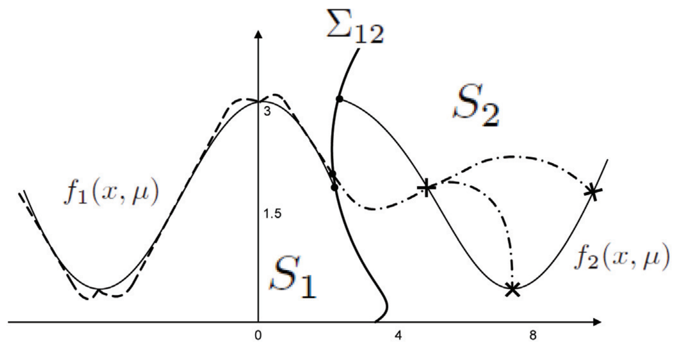
We deliberately said *the best choice* because it has been proven that the affine TS fuzzy system (2.7) can approximate any *smooth* nonlinear function (Tanaka & Wang, 2001). The accuracy of the model will not be affected, whether the TS fuzzy model is obtained from identification of input-output experimental data (Takagi & Sugeno, 1985) or the linearization of the original non-linear function. TS fuzzy identification methods for smooth nonlinear systems are now well-established although there are issues regarding the identification accuracy using input-output data (Tanaka & Wang, 2001).

3. ISSUES AND DIFFICULTIES

Can Current Fuzzy Modelling Approaches Represent a Non-Smooth System?

Sensitive dependence to initial conditions is the signature of nonlinear systems, specially the non-smooth kind. As we saw in section 2.1, a non-smooth system can be much more sensitive to parameter variations; an infinitesimal change in parameter values can change the dynamics of the system from stable period-1 behaviour to a totally unstable chaotic behaviour. In the popular paradigm of chaos theory based on smooth systems, a butterfly flapping its wings in central Asia could start a chain of weather events ending up with a typhoon in a distant place like Florida. Amazingly, if we re-tell the story for a non-smooth nonlinear

Figure 8. Original system (solid) and its Fuzzy approximation (dashed); assuming the existence of fuzzy approximation when losing uniqueness



system, a butterfly flapping its wings in central Asia can instantly create a typhoon in Florida! Of course, real dynamical systems around us can be modelled with both smooth and non-smooth functions. However, that modelling of non-smooth systems to single out colourful, complex nonlinear phenomena can be of vital importance.

Despite numerous attempts to model complex nonlinear system using the TS fuzzy approach, there has been no evidence of its universal suitability to the modelling of discontinuous systems. Although it seems, at first, that the system described by (2.8) can approximate any dynamical function, the equation is in fact structurally and mathematically inadequate for the representation or approximation of any discontinuous function. Let's explain this through a simple example:

Consider the approximation of the following non-smooth system by TS fuzzy system (2.8):

$$\dot{x} = \begin{cases} A\sin(x), & S \text{ is ON}, \quad x \in [-\pi, \pi] \\ A\cos(x), & S \text{ is OFF}, \quad x \in]\pi, 3\pi] \end{cases} \tag{3.1}$$

When the switch S is ON, the function f behaves as a *sine* function and when S is OFF, the function f behaves as a *cosine* function. We select the off-equilibrium linearization method to obtain our fuzzy model from the original function[5]. The idea is to carry out a first-order Taylor expansion at different points θ_i and let the rule base describe the validity of the obtained linear model at each point (Tanaka & Wang, 2001). We try to approximate the original system (3.1) over nine (-2.6,-1.57, 0, 1.57, 2.6, 3.63, 4.66, 5.69, 6.72) linearization point which should normally yield enough modelling accuracy. As envisaged in Figure 8, there is no problem approximating the smooth *sine* function when S in ON, but once the switch goes OFF, the problem arises in approximating the rest of the function.

If we consider the system (3.1) as a *non-smooth* flow function $\Phi(x,t)$, it can be specified by finite set of ordinary differential equations (ODEs), $f_1(x,\mu)$ and $f_2(x,\mu)$, and the state space could be partitioned into different regions S_1 and S_2, each partition being associated with a different set of ODEs since the system behaves differently in each region. Therefore, a formal definition of (3.1) could be:

$$\dot{x} = \begin{cases} f_1(x,\mu), & \text{for } x \in S_1, \quad x \in \Re^n, \mu \in \Re^p \\ f_2(x,\mu), & \text{for } x \in S_2, \quad x \in \Re^n, \mu \in \Re^p \end{cases}$$

where $\cup_i S_i = D \subset R^n, i=1,2$ and each of S_1 and S_2 has a non-empty interior. The intersection Σ_{12} between the closure of the sets S_1 and S_2, $\Sigma_{12} = \bar{S}_1 \cap \bar{S}_2$, is either an $R^{(n-1)}$ dimensional manifold included in the boundaries ∂S_1 and ∂S_2 (called *discontinuity set* or *switching manifold*) or an empty set. Any non-smooth flow Φ has the *degree of smoothness r*, if the first *(r-1)* derivative of Φ with respect to $x \in X$ exist and are continuous at every point $x \in X$, where X is the state space (Di Bernardo, et al., 2007).

We already know that each vector field f_i is smooth in its region S_1, but the problem arises if we try to define a flow $\Phi(x,t)$ for the behaviour of the system as a whole. To overcome this difficulty, there should be a map to define the *transition* just before the switching manifold to just after the switching manifold, otherwise, the function f or the flow Φ are undefined on the switching manifold Σ_{12} at the switching instant.

In a more rigorous mathematical sense, one of the basic methods to conclude the existence and uniqueness of a differential equation with a vector field satisfying only the *local Lipschitz condition* (Khalil & Grizzle, 1996) is if one has some further knowledge of the behaviour of the system (like if the solution stays in a compact subset[6] of the domain where a local Lipschitz condition is satisfied). Generally, the existence of a unique solution of $\Phi(x,t)$ should be guaranteed only over a small interval, say $[t_0, t_0+T]$ where $T > 0$. We can then expand this interval to $[t_0+T, t_0+2T]$ to guarantee the existence of a unique solution of another part of the $\Phi(x,t)$ until we cover the whole state space. Hence, the fuzzy model (2.8) cannot satisfy the Lipschitz property at the point of discontinuity Σ_{12} (which is an instantaneous interval) where the flow $\Phi(x,t)$ is non-existent for the smooth fuzzy system (2.8). Even if we assume the existence of the fuzzy model approximation of the trajectory of the system (3.1) just after the switching manifold, the flow loses it uniqueness as shown by the two simulated trajectories in Figure 8. This is because a fuzzy model of the form (2.8) is essentially looking for a smooth connection between the linearization points (2.6, 3.63), so it will be confused regardless of how many linearization points we choose for maximum accuracy, whether it continues on the smooth trajectory as a continuation of the past interval around the linearization point 4.66 or point 5.69 (visible in Figure 8).

Can Current Fuzzy Engineering Tools Tackle the Stability Analysis of Non-Smooth Systems?

Let's transform the nonlinear system (2.6) into a linear system as follows:

$$\begin{cases} \dot{x}_1 = f_1(x) = x_2 + 3 \\ \dot{x}_2 = f_2(x) = x_1 + x_2 + 3 \end{cases}, \qquad (3.2)$$

This can be described by $\dot{x} = f(x) = Ax + B$ where $A = [0\ 1;1\ 1]$ and $B = [3\ 3]^T$. The vector $x^0 \in R^2$ is called an *equilibrium* of the vector field f if $f(x)=0$. The interesting point is that if any trajectory starts from the equilibrium $x^0 = [0\ -3]^T$ of the linear system (3.2), it will stay there forever. However, if a trajectory starts anywhere near x^0, it may completely diverge from or converge to x^0. The primal theorem of stability states that if the eigenvalues of the state matrix A have negative real part, any trajectory starting from any initial condition, will converge to x_0 and the linear system is (globally) exponentially *stable* (Khalil & Grizzle, 1996). Thankfully there is a way to check the stability of a linear system without solving the system.

Revisiting the nonlinear system (2.6)

$$\begin{cases} \dot{x}_1 = f_1(x) = x_2 + 3 \\ \dot{x}_2 = f_2(x) = x_1^2 + x_2^2 + 3 + u \end{cases},$$

The system have multiple equilibrium points $x_1^0 = (-3, -2\sqrt{3})$ and $x_2^0 = (-3, -2\sqrt{3})$. The main difference between a nonlinear system and a linear one is the fact that a nonlinear system may have multiple equilibria and other invariant sets, which may lead to complex dynamics. A smooth (differentiable) dynamical system can have periodic orbits (limit cycles), non-periodic orbits, invariant torus and chaotic attractors (Kuznetsov, 1998). In a way, the easiest way to study stability is to locally linearize the whole system around the equilibrium point or around the orbit in the case of limit cycles (Kuznetsov, 1998). This linearization is achievable through a Taylor series expansion while neglecting higher order terms. The TS fuzzy modelling approach discussed in section 3.1, is one the best ways to achieve accurate linearization around any desirable point. After linearization, it is possible to check the stability of equilibria as linear systems or the stability of limit cycles by calculating the so-called Floquet multipliers (Kuznetsov, 1998) of the system. Floquet multipliers are the eigenvalues of the linearized matrix around the periodic orbit, which ought to have the real part inside the unit circle for the system to be stable.

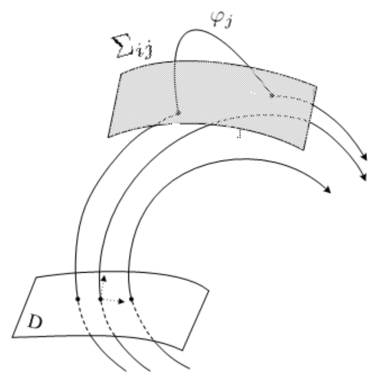
As previously pointed out in section 2.2, most problems in non-smooth dynamical systems are qualitative. In that respect, the qualitative theory of this kind of nonlinear systems demand different notion of stability, called *structural* stability, a concept tied up with *bifurcation theory*. To better understand these concepts, assume the autonomous nonlinear system:

$$\dot{x} = f(x, \alpha), \qquad x \in \Re^n, \qquad \alpha \in \Re^p$$

where *f* is smooth. If through a small change in parameter value, for example the input voltage of a dc to dc converter circuit, the invariant set of the system is transformed to another invariant set, a bifurcation occurs and the system is said to have lost its *structural* stability[7].

A new notion of structural stability is demanded in the case of non-smooth dynamical systems. In this, changes to system parameters may cause, for example, a limit cycle of a non-smooth flow to touch the switching manifold Σ_{12} at a grazing point. In this case, the limit cycle (or in general

Figure 9. In the bifurcation scenario above, a new classification of bifurcation, DIB, is visible when the non-smooth flow hits the switching manifold tangentially at the grazing point and its dynamics qualitatively changed (DIB grazing bifurcation (Di Bernardo, et al., 2007))



invariant set) qualitatively changes with respect to the switching manifold (Figure 9). It therefore gives birth to a new classification of bifurcation, already mentioned, as a discontinuity-induced bifurcation (DIB) (Di Bernardo, et al., 2007). Interesting verities of DIB exist which cannot be found in smooth dynamical system[8]. Sometimes the DIBs of limit cycles can instantly drive a non-smooth system into chaos.

In the Fuzzy control literature, a dominant tool for stability analysis is the *Lyapunov function* technique (Tanaka & Wang, 2001). For non-smooth systems, proving asymptotic stability (even for a classical notion of stability[9]) can be a very demanding task; so the Lyapunov function technique would be potentially a viable solution. However, it is not clear how existing techniques (which can hardly suffice for the stability analysis of equilibria of a non-smooth system) can tackle structural stability problem?

Basically, the Lyapunov technique is all about finding a *common* Lyapunov function, *V(x),* which is positive definite and decreasing along the system trajectory in the whole phase space. For a non-smooth system, the function *V(x)* should be positive definite and decreasing for each of the vector fields defining the system dynamics in each of the phase space regions (Liberzon, 2003). Nevertheless, finding such a Lyapunov function in practice is at best troublesome. Firstly, finding a common Lyapunov function for the whole phase space results in a conservative formulation, this cannot be solved using any analytical or existing numerical method. Secondly, the method is focused on the stability of invariant sets. This second point is of great importance as it stifles the use of Lyapunov theory for proving structural stability. Instead, the dominant stability analysis tool these days is nonlinear discrete modelling, which mainly focuses on examining periodic orbits and their stability by checking the so-called Floquet multipliers of the system (Di Bernardo, et al., 2007; Giaouris, Banerjee, Zahawi, & Pickert, 2008; Leine & Nijmeijer, 2004).

What is the Solution?

One of the main obstacles in studying non-smooth dynamical systems is the lack of rigid procedures for accurate mathematical modelling and numerical simulation. Certainly, wide-spread black-box routines designed to solve smooth dynamical systems cannot be used directly for non-smooth systems since they don't allow for any discontinuous event when the switching manifold Σ_{ij} is crossed. There have been attempts in the past to overcome this problem with two reported methods for numerical simulation of non-smooth system: *event-driven* and *time-stepping* algorithms *(Brogliato, 2000; Brogliato, Ten Dam, Paoli, Genot, & Abadie, 2002)*.

Even though these methods have been applied successfully, depending on the dimension of the specific problem, the modelling and rigorous numerical analysis of non-smooth dynamical systems is still an open research area. Part of the deficiency of the above methods comes from the fact that the modelling, though accurate, ignores the underlying element of this kind of systems, i.e. uncertainty. On the other hand, current fuzzy modelling approaches (section 2.2) can represents a degree of uncertainty, but they are unable to represent the inherent non-smoothness of the original function. Therefore, if we can find a way to somehow modify the fuzzy model (2.7) to embody the discontinuities, this would be an important advance in the modelling of non-smooth dynamical systems.

Asynchronized discrete event systems are a familiar subject in implementing petri-nets (Cassandras, 1993; Hadjicostis & Verghese, 1999). These systems are usually described by:

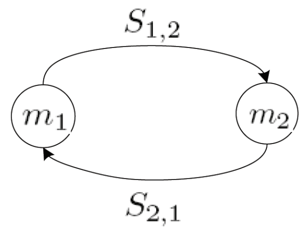
$$m^{+}(t)=\xi(m(t),\sigma(t)),$$

where m is the discrete state variable, σ is the discrete input and ξ is a function describing the change of m. The input σ takes values in a finite set Σ (This Σ should not be mistaken with Σ as discontinuity set or switching manifold), and the elements in Σ are commonly called events. Examples of such events could be "switch S blocking" (case study II) or the "impact with the rigid surface" (case study I). The notation m^{+} conveys the next discrete state of m, or in a more accurate sense, $m^{+}(t)=m(t_{k+1})$ is the successor of $m(t)=m(t_k)$. It is common to drop the time-dependency in m^{+} as it's fairly obvious. Now, a non-smooth dynamical system can be obtained if the corresponding TS fuzzy model consists of interacting continuous and discrete event systems. This means that in addition to interpolating between different continuous sub-systems in the TS fuzzy model (2.7), both continuous and discrete states should interpolate with each other. This is possible through synthesizing a new type of fuzzy model of the form:

$$\begin{cases} \dot{x} = \sum_{j=1}^{l_m} w^{j}(\theta, m)(A^{j}(m)x + B^{j}(m)u + a^{j}(m)) = A(\theta, m)x + B(\theta, m)u + a(\theta, \boldsymbol{m}) \\ m^{+} = \xi(x, m) \end{cases} \quad (3.3)$$

Figure 10.

$$S_{1,2} = \{x \in \Re^n \mid \xi : m_1 \rightarrow m_2\}$$
$$S_{2,1} = \{x \in \Re^n \mid \xi : m_2 \rightarrow m_1\}$$



where $x \in R^n$ is the continuous state, $m \in M = \{m_1, ..., m_N\}$ is a discrete state (N possibly infinite), $A^j(m_i) \in R^{n \times n}$, $B^j(m_i) \in R^n$, $a^j(m_i) \in R$, $w^j: R^n \times M \rightarrow [0,1]$, $j \in I_{lm}$ are continuous weighting functions satisfying $\sum_{j=1}^{lm} w^j(\theta, m) = 1$ and l_m is the number of fuzzy rules. The state space is the Cartesian product $R^n \times M$. The function $\xi: R^n \times M \rightarrow M$ describes the dynamics of the discrete state. The fuzzy model (3.3) basically implies the fuzzy state space F is partitioned to different regions, named $\Omega_q, q \in I_\Delta$, which separated by switching manifold Σ_{ij}. Therefore, each region represents continuous dynamics in the form of set of fuzzy sub-systems

$$\sum_{j \in \{1,2,\ldots\}} w^j(x, m_i)(A^j(m_i)x + B^j(m_i)u),$$
$i \in I_N = \{1,2,...,N\}$,

which we termed *sub-vector field* (K Mehran, D Giaouris, & B Zahawi, 2009) and it is associated with discrete state $m_i \in M$.

For ease of implementation, the switching events can be describe by a number of switch sets $S_{i,k}$, expressed as:

$$S_{i,k} = \{x \in \Re^n \mid m_k = \xi(x, m_i)\}, \quad i \in I_N, k \in I_N, \tag{3.4}$$

The switch set (3.4) is actually employed to represents discontinuity sets or switching manifolds Σ_{ij}. Referring to case study II (section 2.1.2.), if we associate the discrete state m_1 with the dynamics in equation (2.4) when switch S is blocking and the discrete state m_2 with the dynamics in equation (2.4) when switch S is conducting, the switch sets explain the system by the state transition diagram[10] in Figure 10.

As we have the fuzzy model structure (3.3) in hand to represent a non-smooth dynamical system, a method can be devised to study the structural stability or even the stability of equilibria of these systems. Fortunately, there is an immense amount of literature on how TS fuzzy systems can be used with Lyapunov techniques to provide an insight into the system. Unfortunately, in the case of non-smooth dynamical systems, the stability analysis would be implausible without having the *right* model. Moreover, the Lyapunov technique itself hasn't showed any promise in tackling structural stability problems. Nevertheless, in the case of non-smooth system, there are ways of formulating the conditions for studying stability and onset of chaos by acquiring the model structure (3.3). First, however, it is necessary to state the basic stability criteria of TS fuzzy systems modelling smooth dynamics by the following theorem:

Theorem 1: The system $\dot{x} = \sum_{j=1}^{l} w^j(\theta)A^j x$ is asymptotically stable if there exists a positive definite and symmetric matrix P such that:

$$(A^j)^T P + PA^j < 0, \quad \forall j = 1, 2, \ldots, l$$

using the smooth quadratic Lyapunov function $V(x) = x^T Px$, the proof of the theorem fist appeared in (Tanaka & Sugeno, 1992).

As pointed out in section 3.2, the problem of the above formulation (or any other complex formulation) is finding a smooth Lyapunov function over the whole state space. Even for smooth systems, these formulations result in a conservative formulation which in many cases, cannot be numerically solved. After the seminal papers by Paden (Paden & Sastry, 1987; Shevitz & Paden, 1994), it has been shown that the analysis of non-smooth dynamical systems should be based on *discontinuous* or *non-smooth* Lyapunov function. Furthermore, it has been shown, that partitioning the state space into different regions can result in a less conservative (although more complicated) formulation (Johansson, Rantzer, & Arzen, 1999), even in the case of smooth fuzzy systems. Hence, we will base our analysis on *piece-wise* smooth Lyapunov functions, which can be decreasing along the trajectories in *detached* but *flexible* fuzzy state-space regions $\Omega_q, q \in I_\Delta$ (which necessitates that partitioning fulfils $\Omega_1 \cup \ldots \cup \Omega_\Delta = \Omega$ and $\Omega_q \cap \Omega_r = \emptyset, q \neq r$). This simply means that if the trajectory starts from an initial point in region Ω, t_k, $k=1,2,\ldots$, it can pass through to another region on the condition that $t_k < t_{k+1}$. Not all regions are of the same kind, and there should be a general definition for the regions representing switching manifolds. The existence of this kind of region is of vital importance for the structural stability of non-smooth systems, since it defines the *transition* just before the switching manifold to just after the switching manifold in the fuzzy systems by playing the role of the switching manifold $\Sigma_{ij,}$. The formal definition is:

$$\Lambda_{qr} = \left\{ x \in \Re^n \middle| \exists t < t_0, \;\; such\ that \;\; x(t^-) \in \Omega_q,\, x(t) \in \Omega_r \right\}$$

where a set of fuzzy continuous states for which the trajectory $x(t)$ with initial condition $(x_0, m_0) \in F_0$ can pass from Ω_q to Ω_r. It is essential for the regions or the sets Ω_q and Ω_r to be next to each other (hence, we call Λ_{qr} a neighbouring region or set). Λ_{qr} can define any hypersurfaces including switching manifolds with the following conditions:

$$\begin{cases} \Sigma_{qr} = 0 \\ \sum_{j \in \{1,2,\ldots\}} w^j(x, m_i)(A^j(m_i)x + B^j(m_i)u) \cdot \nabla\Sigma_{qr} < 0 \end{cases}$$

Since the trajectory should ultimately pass (or in the case of neighbouring regions map) from one region to another, we define another kind of region as:

$I_\Lambda = \{(q,r) | \Lambda_{qr} \neq \Phi\}$

which is a set of tuples indicating that there is at least one point for which the trajectory passes from Ω_q to Ω_r.

For each fuzzy region, a Lyapunov energy function V_q:***cl*** $\Omega_q^x \rightarrow$ R, $q \in I_\Delta$, (***cl*** denotes the closure of a set, which is the smallest closed set containing a set) is assumed to be continuously differentiable on ***cl*** $\Omega_q^x \rightarrow$ R, $q \in I_\Delta$. The time derivative of $V_q(x)$ is expressed as:

$$\dot{V}(x) = \sum_{j=1}^{l_m} w^j(\theta, m) \frac{\partial V_q(x)}{\partial t}(A^j(m) + B^j(m)), \qquad (x, m) \in \Omega_q$$

The overall Lyapunov function $V(x)$, composed of local $V_q(x)$, is *discontinuous* at neighbouring regions Λ_{qr}, $(q,r) \in I_\Delta$ and with the assumption of $t_k < t_{k+1}$ for every trajectory with an initial point in any region, is *piecewise* with respect to time.

Searching for a Lyapunov function, except in some simple cases, is a very complicated analytical task. The other option is to numerically search for the function, which necessitate formulating the stability conditions as Linear Matrix Inequalities (LMI) which may be solved by interior point methods (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994). Therefore, the first stage here is to define each fuzzy state-space partitions by positive (quadratic) functions. A so-called *S-procedure* technique makes this definition possible by substituting confined conditions with unconfined conditions (Boyd, et al., 1994). To explain the substitution, the procedure is first elaborated in general terms and then applied to the confined conditions of the forthcoming stability theorem (Mehran, et. al., 2009).

$Q_0,\dots, Q_s$ being the quadratic function of the variable $x\in R^n$ of the form:

$$Q_k(x) = x^T Z_k x + 2c_k^T x + d_k, \quad k = 0,\dots,s, \tag{3.5}$$

where $Z_k = Z_k^T$, we can impose the following condition:

$$Q_0(x) \geq 0 \quad \text{in the region} \quad \{x \in \Re^n \mid Q_k(x) \geq 0,\ k \in I_S\} \tag{3.6}$$

The first two conditions in the stability theorem can be formulated in the form:

$$Q_0(x) \geq 0 \quad \text{in the region} \quad \Omega_q^x, \tag{3.7}$$

where $Q_0(x)\geq 0$ is the corresponding condition in the region. By searching the quadratic functions $Q_0(x)\geq 0$, $k\in I_S$ under the condition $\Omega \in \{x \in \Re^n \mid Q_k(x) \geq 0,\ k \in I_S\}$, it is clear that if condition (3.6) is fulfilled, the condition (3.7) will be automatically fulfilled too. The extreme case would be letting the condition $Q_0(x)\geq 0$ be valid in the entire state space, although this should normally be averted since this conservative formulation may end up not finding a solution for the original condition (3.7). By including Ω in a region specified by the quadratic function, the confined condition (3.6) can be substituted by an unconfined condition through the following Lemma:

Lemma (Boyd, et al., 1994)**:** If there exists $\delta_k\geq 0$, $k\in I_S$, such that

$$\forall x \in \Re^n,\ Q_0(x) \geq \sum_{k=1}^{s} \delta_k Q_k(x)$$

then (3.6) holds. Therefore, by introducing additional variables $\delta_k\geq 0$, $k\in I_S$, the condition (3.7) can be formulated as the following LMI condition:

$$x^T \begin{bmatrix} Z_0 & c_0 \\ c_0^T & d_0 \end{bmatrix} x \geq \sum_{k=1}^{s} \delta_k x^T \begin{bmatrix} Z_0 & c_0 \\ c_0^T & d_0 \end{bmatrix} x \tag{3.8}$$

If the fuzzy region partitions Ω_q, $q\in I_\Delta$ are represented by a single quadratic form, conditions (3.6), (3.7) and the condition of Lemma will be equivalent. We already know that the neighbouring regions are given by the switching manifold. Hence, they can be represented by $Q_k(x) = x^T Z_k x + 2c_k^T x + d_k = 0,\ k \in I_S$, where there is no constrain such as $\delta_k\geq 0$, $k\in I_S$ in Lemma, since Lemma holds regardless of the sign of δ_k[11].

In order to incorporate the candidate Lyapunov functions into an LMI problem, they should be defined as piecewise quadratic with the following structure:

$$(x,m) \in \Omega_q,\quad V_q(x) = \pi_q + 2p_q^T x + x^T P_q x$$

where

$$\pi_q \in \Re,\ p_q \in \Re^n,\ P_q = P_q^T \in \Re^n \times \Re^n,\ q \in I_\Delta$$

Therefore the overall d*iscontinuous* Lyapunov function candidate *V(x)* is defined as:

$$V(x) = V_q(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_q & p_q \\ p_q^T & \pi_q \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \tilde{x}^T \tilde{P}_q \tilde{x}, \quad (x,m) \in \Omega_q \tag{3.9}$$

Now, by describing $Q_k(x)$ in its matrix form:

$$Q_k(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Z_k & c_k \\ c_k^T & d_k \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \tilde{x}^T \tilde{Z}_k \tilde{x}, \quad k = 0,\dots,s, \tag{3.10}$$

and by assuming that the fuzzy region partitions Ω_q, $q \in I_\Delta$ are represented by the LMI form $Q_k^q(x) \geq 0,\ k \in I_{Sq}$, that the neighbouring region Λ_{qr}, $(q,r) \in I_\Lambda$ is represented by the LMI form of $Q_k^{qr}(x) \geq 0,\ k \in I_{Sqr}$ and by substituting the conditions given in different fuzzy regions by the LMI condition (3.8), the stability conditions can be formulated as an LMI problem (Mehran, et. al., 2009) as follows:

LMI problem: If there exists $\tilde{P}_q$, $q \in I_\Delta$, constants $\alpha > 0$, $\mu_k^q \geq 0$, $\nu_k^{qij} \geq 0$, $\eta_k^{qr} \geq 0$ and a solution to min β subject to the following conditions:

1.

$$Q_0(x) \geq \tilde{P}_q - \alpha\tilde{I} + \sum_{k=1}^{s_q} \mu_k^q \begin{bmatrix} Z_k^q & c_k^q \\ (c_k^q)^T & d_k^q \end{bmatrix} \geq 0, \qquad q \in I_\Delta$$

2.

$$Q_0(x) \geq \beta\tilde{I} + \sum_{k=1}^{s_q} \mu_k^q \begin{bmatrix} Z_k^q & c_k^q \\ (c_k^q)^T & d_k^q \end{bmatrix} - \tilde{P}_q \geq 0, \qquad q \in I_\Delta$$

3.

$$Q_0(x) \geq -\left((\tilde{A}^j)^T \tilde{P}_q + \tilde{P}_q \tilde{A}^j + \sum_{k=1}^{s_{qij}} \nu_k^{qij} \begin{bmatrix} Z_k^q & c_k^q \\ (c_k^q)^T & d_k^q \end{bmatrix} + \gamma\tilde{I} \right) \geq 0, \qquad (q,i,j) \in I_\Delta$$

4.

$$Q_0(x) \geq \tilde{P}_q - \tilde{P}_r - \sum_{k=1}^{s_{qr}} \eta_k^{qr} \begin{bmatrix} Z_k^{qr} & c_k^{qr} \\ (c_k^{qr})^T & d_k^{qr} \end{bmatrix} = 0, \qquad (q,r) \in I_\Delta$$

then, the fixed point is exponentially stable in the sense of Lyapunov[12], where $\tilde{I}$ is the matrix [1 0;0 0].

In the first and second conditions, α and β are constants but originally represent a class K function

$$\alpha(\|x\|),\ \alpha : \Re^+ \to \Re^+ \text{ and } \beta(\|x\|),\ \beta : \Re^+ \to \Re^+$$

(the definition of a class K function can be found in (Khalil & Grizzle, 1996)). $\gamma > 0$ is simply a scalar constant. As the forth condition is satisfied on the neighbouring region Λ_{qr} (or the switching manifold) the condition for fuzzy partitions should be given by $Q_k(x) = 0$, $k \in I_S$, where each $Q_k(x)$ has the form (3.6).

The LMI problem can predict the onset of bifurcation or instability leading to chaos for the non-smooth dynamical system represented by the fuzzy model (3.3) through the fixed point of the system's trajectory. The fixed point of the poincaré map is the intersection of the limit cycle with the map. The above stability conditions investigate when the limit cycle loses its stability by checking if the Lyapunov function candidate (3.5) is decreasing along the trajectories in each local fuzzy partition Ω_q, from local partition Ω_q to Ω_k within a fuzzy sub-vector field and most importantly on the neighbouring regions Λ_{qr}, which is the point at which the energy function becomes discontinuous.

To prove the applicability of the theoretical analysis, we applied the TS fuzzy model structure (3.3) and (3.4) to the buck converter (Mehran, Giaouris, & Zahawi, 2008) described in case study II to see if the edge of the bifurcation will be successfully detected by this new method. Figure 11 shows that the bifurcation occurs when the input voltage V_{in} is changed from $24V$ to $25V$. This can also be predicted by the LMI problem showing an infeasible solution when the input voltage is changed to $25V$ but being exponentially stable for $V_{in} \leq 24V$ with the optimal value of $\beta = 2.4962$ (Mehran, et al., 2008).

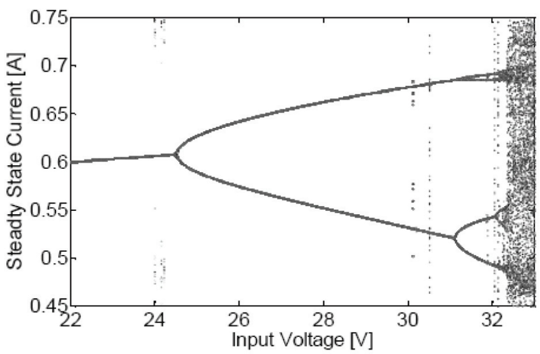
4. CONCLUSION AND FUTURE RESEARCH

The proposed TS fuzzy modelling method is able to approximate non-smooth dynamical systems with varying degrees of smoothness. For instance, it is possible to include state jumps in the dynamics of the model (3.3) by another function $\chi: D_\chi \to R^n$, where:

$$x^+ = \chi(x, m_i)$$

and the set of states where the state jump takes place for different discrete states can be equivalently explained by the following sets:

Figure 11. The bifurcation diagram of the TS fuzzy model of the converter is well matched with the original bifurcation diagram shown previously. It also shows how the stable limit cycle of the converter loses its stability to a period-doubling bifurcation



$$J_i = \left\{ x \in \Re^n \mid x^+ = \chi(x, m_i) \right\}, \qquad i \in I_N$$

Therefore, by letting switch sets $S_{i,j}$ and J_i concur at certain fuzzy continuous states *($x \in F$)*, a jump can occur at both discrete and continuous states. Moreover, by employing *non-determinism* in discrete event systems and by defining another discrete state, it is possible to represent *sliding mode* behaviour in the model (3.3), which is typical of mechanical relay systems.

One of the difficulties when attempting to simulate a NSDS is that existing *time-stepping* or *event-driven* modelling methods can only be implemented using special purpose software packages. No single software platform is yet available to accurately simulate and study the performance characteristics of these systems. Commercial algorithms have not yet been designed to allow robust numerical computations of all NSDS with different degrees of smoothness; consequently these are computationally expensive options. The proposed fuzzy method may allow the possibility of designing algorithms that do not require special-purpose software platforms. More importantly the new method, although not complicated, is able to approximate the uncertainty inherent in non-smooth systems that tends to be neglected by other available methods for the modelling of discontinuous systems.

Fuzzy engineering has unequivocally shown (at times with other AI-based methods) that it is a powerful tool for system identification. The proposed model structure (3.3) and (3.4) can be especially valuable when obtained from experimental input-output data and in situations when access of the original function is impossible. In this case, the Lyapunov framework proposed for stability analysis is a very useful tool in predicting the onset of dynamic changes (DIBs and chaos) in non-smooth systems. The main reason for this is that the Lyapunov analysis attempts to investigate the stability of the system in an indirect way while other approaches directly target the problem, which in the case of high-order complex non-smooth systems, may result in an incomputable solution.

The last two decades have witnessed considerable efforts in the attempt to control chaotic systems. Chaos control strategies are still difficult to implement and in many cases, still in their theoretical stages. This analysis has shown that a novel TS fuzzy model-based control scheme can be devised, using the LMI formulation, to stabilize the system to its professed period-1 behaviour. One of the design approaches based on the proposed methodology here, is expounded in (Mehran, et. al., 2009).

However, there are some issues that need to be addressed. First, the proposed stability conditions cannot pinpoint the higher period-doubling bifurcation. More sophisticated LMI formulations are needed to specify the kind of instability, which could be an unstable limit cycle, interspersed by some completely chaotic periods. In this respect, the existing interior point methods should be enhanced to better solve those LMI problems.

Second, fuzzy state-space partitioning does not always lead to an actual solution. In some cases, a high number of region partitions, although representing the region more accurately, can end up in formulations which are misleadingly infeasible. Therefore, there is always a delicate balance to strike between high accuracy and complexity. A general guideline for state-space partitioning needs to be developed with some urgency.

Most complex systems in reality, including many Kansei applications where *uncertainty* and *discontinuity* are two essential elements of data gathering and processing, are composed of different nonlinear systems networked together. The behaviour of these systems could be far more complex if they are networks of non-smooth systems. The application of fuzzy engineering in modelling the inner interaction of these complex systems and quantifying their substantial uncertainty is an important subject for future research.

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ENDNOTES

1 Berkeley Initiative for Soft Computing (BISC) programme: http://www-bisc.cs.berkeley.edu/BISCProgram/default.htm

2 "Toward a Generalized Theory of Uncertainty (GTU)—An Outline", article by Lotfi A. Zadeh to appear on information sciences, January 20, 2005

3 Private conversation with author, London June 2009.

4 Webster's defines uncertainty as the quality or state of being uncertain. An uncertain issue is something not exactly known; not determined, certain, or established; a contingency.

5 A detail explanation can be found in (Tanaka & Wang, 2001) and good collection of ex-

amples in (Palm, Driankov, & Hellendoorn, 1997) and (Johansen, Hunt, Gawthrop, & Fritz, 1998).

6 A subset of Euclidean space R^n is called a compact set if it is closed and bounded. For example, in R, the closed unit interval [0, 1] is compact, but the set of Z is not (it is not bounded) and neither is the half-open interval [0, 1) (it is not closed). However, generally, a topological space called compact if each of its open covers has a finite sub-covers. Otherwise it is called non-compact.

7 Bifurcation theory can be further investigated in (Kuznetsov, 1998).

8 Bifurcations occurring in smooth systems are termed *smooth bifurcations* because they have nothing to do with switching manifolds.

9 Stability of equilibria whose eigenvalues are located in the right-half plane.

10 It is noteworthy that $S_{i,k}$ represents, in general, the set of all values of x that the switching conditions can apply. Therefore, putting them directly on state transition diagram is just for ease of illustration, though, according to formal definition, we should put the switching condition directly on top of each transition arrow.

11 If in any case, the switching manifold cannot be represented by $Q_k(x)=0$, $k \in I_S$, there is a possibility to represent such manifold by larger switch regions. However, this may end up in a conservative LMI formulation.

12 The theoretical proof of the theorem can be found in (Mehran, et. al., 2009).